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A pedal triangle inequality with the exponents

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Abstract

In this paper we establish a new pedal triangle inequality with the exponent variables. Some related interesting conjectures verified by the computer are put forward.

Keywords: *pedal triangle, interior point, exponent, transformation.*

1 Introduction

Let P be an interior point of the $\triangle ABC$ and let D, E, F denote the feet of the perpendiculars from P to sidelines BC, CA, AB. Denote the semiperimeter, area, circumradius and inradius of the $\triangle ABC$ by s, S, R, r, and denote the area, circumradius and inradius of the pedal $\triangle DEF$ by S_p, R_p, r_p respectively. Following the notation of [1] and [2], put $BC = a, CA = b, AB = c, PA = R_1, PB = R_2, PC = R_3, PD = r_1, PE = r_2, PF = r_3$ (see Figure 1).

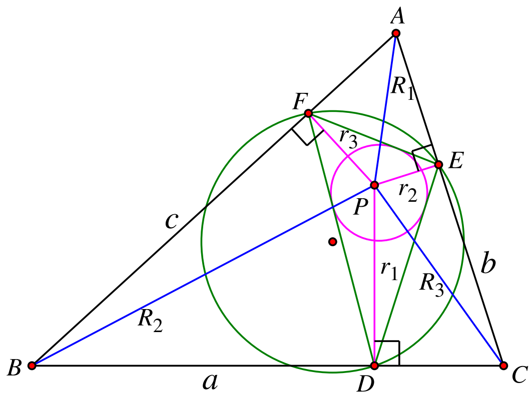

Figure 1

From the published literatures (see, e.g., [1], [2]), there are few inequalities involving triangles and its an interior point with exponential variables. In [3], the author established several geometric inequalities related to the pedal triangle. One of the results is the following:

$$\frac{1}{R_1^k}+\frac{1}{R_2^k}+\frac{1}{R_3^k}\geq\frac{1}{(2R_p)^k}+\frac{2}{R^k}, \tag{1}$$

where the exponent k satisfies $k \geq 1$. The equality holds if and only if $\triangle ABC$ is equilateral and P is its center.

The author also conjectured that inequality (1) is reverse when $-1 \leq k < 0$. In other words, the following inequality

$$R_1^k+R_2^k+R_3^k\leq 2R^k+(2R_p)^k \tag{2}$$

holds for $0 < k \leq 1$. Recently, Wang Zhen [4] has proved the special case $k = 1$:

$$R_1+R_2+R_3\leq 2(R+R_p). \tag{3}$$

It is easy to prove that

$$4r_p \leq R. \tag{4}$$

Inequalities (3) and (4) prompt the author to find that

$$R_1+R_2+R_3\geq 2R_p+8r_p. \tag{5}$$

That is to say, if we change R in (3) by a smaller value $4r_p$, then the inequality is reverse. Generally, we have the following conclusion:

Theorem 1.1 *If $k \geq 1$ be a real number, then for any interior point P of the $\triangle ABC$ holds:*

$$R_1^k+R_2^k+R_3^k\geq (2R_p)^k+2(4r_p)^k. \tag{6}$$

If $k = -1$, then the inequality is reverse. The equalities hold if and only if $\triangle ABC$ is equilateral and P is its center.

From the theorem we have the following reciprocal type inequality:

$$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\leq\frac{1}{2}\left(\frac{1}{R_p}+\frac{1}{r_p}\right). \tag{7}$$

It seems to be difficult to prove (7) directly. We will give a simple proof by using a known inequality involving two triangles.

Incidentally, it does not discriminate strength or weakness between the beautiful linear inequality (5) and the famous Erdös-Mordell inequality (see, e.g., [1], [2], [5], [6], [7]):

$$R_1+R_2+R_3\geq 2(r_1+r_2+r_3), \tag{8}$$

which is given a new proof recently by the author in [5].

The purpose of this paper is to prove Theorem 1.1 and put forward some related open problems (conjectures).

2 Proof of Theorem 1.1

Lemma 2.1 *For any $\triangle ABC$ and positive real numbers x, y, z, we have*

$$\frac{s-a}{x}+\frac{s-b}{y}+\frac{s-c}{z} \geq \frac{s(xa+yb+zc)}{yza+zxb+xyc}, \tag{9}$$

with equality if and only if $x = y = z$.

In [3], the author has pointed out that inequality (9) can be deduced from Klamkin's the polar moment of the inertia inequality (see [2], [8], [9], [10]).

Lemma 2.2 *If $k \geq 1$ is a real number, then we have*

$$\frac{1}{r_1^k}+\frac{1}{r_2^k}+\frac{1}{r_3^k} \geq \frac{2}{r^k}+\frac{S^k}{(2RS_p)^k}, \tag{10}$$

with equality if and only if P is the incenter of $\triangle ABC$.

Proof. We first prove the case $k = 1$. Putting $x = r_1, y = r_2, z = r_3$ in (9), then using the following two identities:

$$ar_1 + br_2 + cr_3 = 2S, \tag{11}$$

$$ar_2r_3 + br_3r_1 + cr_1r_2 = 4RS_p, \tag{12}$$

we get

$$\frac{s-a}{r_1}+\frac{s-b}{r_2}+\frac{s-c}{r_3} \geq \frac{sS}{2RS_p}. \tag{13}$$

Hence

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} \geq \frac{1}{s}\left(\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}\right)+\frac{S}{2RS_p} \geq \frac{2}{r}+\frac{S}{2RS_p},$$

where we used the inequality:

$$\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3} \geq \frac{2s}{r} \tag{14}$$

which is the special case $k = 1$ of inequality (17) below. This completes the proof of the case $k = 1$ of (10).

Next, we prove the case $k > 1$.

When $k > 1$, using the weighted power means inequality, we have that

$$\left[\frac{(s-a)r_1^{-k}+(s-b)r_2^{-k}+(s-c)r_3^{-k}}{(s-a)+(s-b)+(s-c)}\right]^{\frac{1}{k}}$$
$$\geq \frac{(s-a)r_1^{-1}+(s-b)r_2^{-1}+(s-c)r_2^{-1}}{(s-a)+(s-b)+(s-c)}.$$

Hence

$$\frac{s-a}{r_1^k}+\frac{s-b}{r_2^k}+\frac{s-c}{r_3^k}\geq\frac{1}{s^{k-1}}\left(\frac{s-a}{r_1}+\frac{s-b}{r_2}+\frac{s-c}{r_3}\right)^k, \tag{15}$$

with equality if and only if $r_1=r_2=r_3$, namely P is the incenter of $\triangle ABC$.

From (13) and (15) we get

$$\frac{s-a}{r_1^k}+\frac{s-b}{r_2^k}+\frac{s-c}{r_3^k}\geq s\left(\frac{S}{2RS_p}\right)^k, \tag{16}$$

where $k>1$. On the other hand, from [1] (P_{285}), we have the following inequality:

$$\frac{a}{r_1^k}+\frac{b}{r_2^k}+\frac{c}{r_3^k}\geq\frac{2s}{r^k}, \tag{17}$$

where $k>0$ or $k<-1$. The equality holds if and only if P is the incenter of $\triangle ABC$. Adding up (16) and (17) then dividing both sides by s, we see that inequality (10) holds for $k>1$.

Combing with the arguments of the two cases above, (9) holds for $k\geq 1$. It is easy to know that the equality in (10) holds only when P is the incenter of the $\triangle ABC$. The proof of Lemma 2.2 is complete.

Lemma 2.3 *If the following inequality:*

$$f(a,b,c,R_1,R_2,R_3,r_1,r_2,r_3)\geq 0 \tag{18}$$

holds for any interior P of the $\triangle ABC$, then the inequality holds by the following K transformation:

$$\begin{aligned}&(a,b,c,R_1,R_2,R_3,r_1,r_2,r_3)\\&\to\left(\frac{aR_1}{2r_2r_3R},\frac{bR_2}{2r_3r_1R},\frac{cR_3}{2r_1r_2R},\frac{1}{r_1},\frac{1}{r_2},\frac{1}{r_3},\frac{1}{R_1},\frac{1}{R_2},\frac{1}{R_3}\right).\end{aligned}$$

The above K transformation is called reciprocation transformation (see [2], [3], [7], [11]).

Lemma 2.4 [12] *Let S_a,S_b,S_c be the area of $\triangle PBC,\triangle PCA,\triangle PAB$ respectively. Then we have*

$$S_aR_1+S_bR_2+S_cR_3\geq 4RS_p, \tag{19}$$

with if and only if P is the incenter of the $\triangle ABC$.

Lemma 2.5 *For any $\triangle ABC$ and $\triangle A'B'C'$ with sides a',b',c' and circumradius R', we have*

$$\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\leq R'\left(\frac{1}{R}+\frac{1}{r}\right), \tag{20}$$

with equality if and only if the two triangles are both equilateral.

The author [13] has given the following equivalent version of (20):

$$\frac{\sin A'}{\sin A}+\frac{\sin B'}{\sin B}+\frac{\sin C'}{\sin C}\le\frac{R+r}{r} \tag{21}$$

and its generalization:

$$\sqrt{r_2r_3}\frac{\sin A'}{\sin A}+\sqrt{r_3r_1}\frac{\sin B'}{\sin B}+\sqrt{r_1r_2}\frac{\sin C'}{\sin C}\le R+r. \tag{22}$$

In addition, inequality (20) was also given by D.Veljan and S.H.Wu in [14].

We now prove our Theorem 1.1.

Proof. We first prove the case $k\ge 1$.

In [3], the author has pointed out the following relations under K transformation:

$$S\to\frac{S}{2r_1r_2r_3R}, R\to\frac{R_1R_2R_3}{4r_1r_2r_3R}, S_p\to\frac{S}{2R_1R_2R_3R_p}.$$

According to these relations, it is easily known that

$$\frac{S}{RS_p}\to 4R_p \tag{23}$$

under K transformation. In addition, using $r=\frac{S}{s}$ we have that

$$r\to\frac{S}{S_aR_1+S_bR_2+S_cR_3} \tag{24}$$

under the same transformation.

If we apply K transformation to inequality (10) of Lemma 2.2, then make using of (23) and (24), we obtain

$$R_1^k+R_2^k+R_3^k\ge\frac{2(S_aR_1+S_bR_2+S_cR_3)^k}{S^k}+(2R_p)^k.$$

From this and inequality (19), we immediately obtain

$$R_1^k+R_2^k+R_3^k\ge(2R_p)^k+2\left(\frac{4RS_p}{S}\right)^k, \tag{25}$$

where $k\ge 1$. Again, noticing the following known inequality (see [12]):

$$\frac{S_p}{r_p}\ge\frac{S}{R}, \tag{26}$$

the required inequality (6) follows from (25) at once.

Now, we prove the case $k=-1$.

Exchanging the two triangles in (20), then we get

$$\frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'} \leq R\left(\frac{1}{R'} + \frac{1}{r'}\right). \tag{27}$$

(r' is the circumradius of $\triangle A'B'C'$)Namely,

$$\frac{\sin A}{a'} + \frac{\sin B}{b'} + \frac{\sin C}{c'} \leq \frac{1}{2}\left(\frac{1}{R'} + \frac{1}{r'}\right). \tag{28}$$

If we assume that $\triangle A'B'C'$ just is the pedal $\triangle DEF$, then

$$\frac{\sin A}{EF} + \frac{\sin B}{FD} + \frac{\sin C}{DE} \leq \frac{1}{2}\left(\frac{1}{R_p} + \frac{1}{r_p}\right). \tag{29}$$

Noticing that $EF = R_1 \sin A$ etc., inequality (7) follows from the above inequality immediately. It is easily concluded that the equalities of (6) and (7) hold if and only if $\triangle ABC$ is equilateral and P is its center (We omit the details). This completes the proof of Theorem 1.1.

3 Open Problems

In this section we propose some related conjectures.

For Theorem 1.1, we put forward the following two conjectures checked by the computer:

Conjecture 3.1 *If* $0 < k < 1$*, then inequality (6) holds.*

Conjecture 3.2 *If* $0 < k < 1$*, then we have*

$$\frac{1}{R_1^k} + \frac{1}{R_2^k} + \frac{1}{R_3^k} \leq \frac{1}{(2R_p)^k} + \frac{2}{(4r_p)^k}. \tag{30}$$

In [3], the author conjectured that the inequality of Lemma 2.2 holds for $-1 \leq k < 0$. Now, we have known this conjecture is not valid, and then the following related conjecture is brought up:

Conjecture 3.3 *If* $0 < k < 1$*, then inequality (10) holds.*

If the above conjecture holds true, from the proof of Theorem 1.1, it is easily seen that Conjecture 3.1 holds. In addition, if Conjecture 3.3 is true then we will know that the preceding inequality (1) and “ The Five Circles Inequality ” in [3]:

$$R_a^k + R_b^k + R_c^k \geq R^k + 2^{k+1} R_p^k \tag{31}$$

(where R_a, R_b, R_c are the circumradius of $\triangle PBC, \triangle PCA, \triangle PAB$ respectively) also hold for $0 < k < 1$ (the case when $k \geq 1$ is proved in [3]). There is one thing we should pay attention to: When $0 < k < 1$, it is easy to prove that inequality (15) holds reversely. This means that if we want to prove Conjecture 3.3 then we have to use other methods different from the proof of Lemma 2.2.

From the proved inequality (3) and Erdös-Mordell inequality (8), we have the following beautiful inequality:

$$r_1 + r_2 + r_3 \leq R_p + R. \tag{32}$$

Considering its exponential generalization, we propose

Conjecture 3.4 *If* $0 < k \leq 1$*, then we have*

$$r_1^k + r_2^k + r_3^k \leq R_p^k + 2^{1-k} R^k. \tag{33}$$

If $k < 0$*, then the inequality is reverse.*

The case $k < 0$ of Conjecture 3.4 is just equivalent to

$$\frac{1}{r_1^k} + \frac{1}{r_2^k} + \frac{1}{r_3^k} \geq \frac{1}{R_p^k} + \frac{2^{k+1}}{R^k}, \tag{34}$$

where $k > 0$. This inequality and Euler inequality $R_p \geq 2r_p$ in the pedal triangle DEF inspire the author to pose the following conjecture:

Conjecture 3.5 *If* $\triangle ABC$ *is an acute-angled triangle and* $0 < k \leq 4$*, then we have*

$$\frac{1}{r_1^k} + \frac{1}{r_2^k} + \frac{1}{r_3^k} \geq \frac{1}{(2r_p)^k} + \frac{2^{k+1}}{R^k}. \tag{35}$$

Remark 3.1 *The triangles unexplained in this note are all arbitrary.*

In [3], the author has proved inequality (10) is also valid for $k \leq -1$. In other words, we have that

$$r_1^k + r_2^k + r_3^k \geq 2r^k + \frac{(2RS_p)^k}{S^k}, \tag{36}$$

where $k \geq 1$. Noticing that

$$\frac{2RS_p}{S} \geq \frac{r_1 r_2 r_3}{r^2}, \tag{37}$$

which is equivalent to (14) by (12). So we have the following interesting inequality among r_1, r_2, r_3 and r:

$$r_1^k + r_2^k + r_3^k \geq 2r^k + \frac{(r_1 r_2 r_3)^k}{r^{2k}}, \tag{38}$$

($k \geq 1$)with equality if and only if P is the incenter of the $\triangle ABC$. This inequality leads us to find the the following dual acute triangle inequality:

Conjecture 3.6 *If $\triangle ABC$ is acute-angled triangle and $k \geq 1$, then*

$$R_1^k + R_2^k + R_3^k \geq 2R^k + \frac{(R_1R_2R_3)^k}{R^{2k}}, \tag{39}$$

with equality if and only if P is the circumcenter of $\triangle ABC$.

The above conjecture can also be stated as follows: Suppose $k \geq 1$, and $\lambda_1 = \left(\frac{R_1}{R}\right)^k, \lambda_2 = \left(\frac{R_2}{R}\right)^k, \lambda_3 = \left(\frac{R_3}{R}\right)^k$, then the the following inequality:

$$\lambda_1 + \lambda_2 + \lambda_3 - \lambda_1\lambda_2\lambda_3 \geq 2 \tag{40}$$

holds for any interior point P of the acute-angled $\triangle ABC$.

Finally, we put forward three interesting conjectures for Cevian triangles, which are similar to the previous several inequalities.

Let P be an interior point of $\triangle ABC$ and let AP, BP, CP cut BC, CA, AB at L, M, N respectively (see Figure 2). Put $PL = e_1, PM = e_2, PN = e_3$. Denote by R_a, R_b, R_c the circumradius of $\triangle PBC, \triangle PCA, \triangle PAB$ and denote by R_q, r_q the circumradius, inradius of the Cevian triangle LMN respectively.

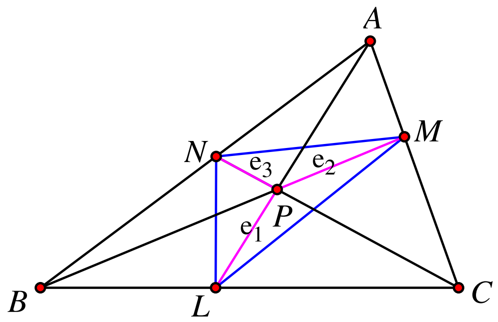

Figure 2

For inequality (31), we present the following similar conjecture:

Conjecture 3.7 *If $k \geq \frac{3}{4}$, then we have*

$$R_a^k + R_b^k + R_c^k \geq (2R_q)^k + 2^{k+1}r^k. \tag{41}$$

For inequality (34), we propose the following dual conjecture:

Conjecture 3.8 *If $k > 0$, then we have*

$$\frac{1}{e_1^k} + \frac{1}{e_2^k} + \frac{1}{e_3^k} \geq \frac{1}{R_q^k} + \frac{2^{k+1}}{R^k}. \tag{42}$$

From (26) and (36), we obtain inequality:

$$r_1^k + r_2^k + r_3^k \geq 2r^k + (2r_p)^k, \tag{43}$$

where $k \geq 1$. The similar conjecture inequality about for Cevian triangles is

Conjecture 3.9 *If $k \geq 2.1$, then we have*

$$e_1^k + e_2^k + e_3^k \geq 2r^k + (2r_q)^k. \tag{44}$$

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