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A common random fixed point theorem for six weakly compatible mappings in Hilbert spaces

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Abstract

In this paper, we obtain a common random fixed point theorem for six weakly compatible random operators defined on a nonempty closed subset of a separable Hilbert space under some conditions.

Keywords: *Common random fixed point, Separable Hilbert space, Six random mappings, Weakly compatible mappings.*

1 Introduction

Random fixed point theorems are stochastic gereralization of classical fixed point theorems. Random fixed point theorems for contraction mappings on separable complete metric spaces have been proved by several authors (See e.g. Spacek [22], Hans [9],[10], Bharucha-Reid [7], Itoh [11], Mukherjee [17], Tan and Yuan [23]) and many others.
In 1982, Sessa [20] introduced the notion of weakly commuting mappings. Jungck [12] defined the notion of compatible mappings to generalize the concept of weak commutativity and showed that weakly commuting mappings are compatible but the converse is not true [12] and a number of fixed point theorems have been obtained by various authors utilizing this notion ([13], [14], [16], [18], [19], [21]). Jungck further weakens the notion of compatibility by introducing the notion of weak compatibility and in [15], Jungck and Rhoades

further extended weak compatibility to the setting of single-valued and multi-valued maps.
Afterwards, Beg [1], [2], Beg and Shahzed [5], [6] studied the structure of common random fixed points and random coincidence points of a pair of compatible random operators and proved the random fixed points theorems for contraction random operators in Polish spaces. Some random fixed point theorems for weakly compatible random operators under generalized contractive conditions are proved by Beg [3], Beg and Abbas [4] and others.
In continuation of these results, motivated and inspired by the contraction condition by Ćirić [8], we obtain a common random fixed point for weakly compatible six mappings on a nonempty closed subset of a separable Hilbert space H.

2 preliminaries

Let (Ω, Σ) be a measurable space, H stands for a separable Hilbert space and C a nonempty closed subset of H.
A mapping $\xi : \Omega \to C$ is called measurable if $\xi^{-1}(B \bigcap C) \in \Sigma$ for every Borel subset B of H.
A mapping $T : \Omega \times C \to C$ is said to be random mapping if for each fixed $x \in C$, the mapping $T(., x) : \Omega \to C$ is measurable.
A measurable mapping $\xi : \Omega \to C$ is called a random fixed point of the random mapping $T : \Omega \times C \to C$ if $T(w, \xi(w)) = \xi(w)$ for each $w \in \Omega$.

Definition 2.1 *[15] Let H be a separable Hilbert space. Random operators $S, T : \Omega \times H \to H$ are weakly compatible if $T(w, \xi(w)) = S(w, \xi(w))$, for some measurable mappings ξ, then $T(w, S(w, \xi(w))) = S(w, T(w, \xi(w)))$ for every $w \in \Omega$.*

Condition (A) *Six random mappings E, F, S, T, A and $B : \Omega \times C \to C$, where C is a nonempty closed subset of a separable Hilbert space H are said to satisfy condition **A** if*

$$\begin{aligned}
\|E(w,x) - F(w,y)\|^2 &\leq \alpha(w)\max\{\|S(w,A(w,x)) - T(w,B(w,y))\|^2 \\
&\quad , \|S(w,A(w,x)) - E(w,x)\|^2, \|T(w,B(w,y)) - F(w,y)\|^2 \\
&\quad , \tfrac{\|S(w,A(w,x)) - F(w,y)\|^2 + \|T(w,B(w,y)) - E(w,x)\|^2}{2}\} \\
&\quad + \beta(w)\max\{\|S(w,A(w,x)) - E(w,x)\|^2, \|T(w,B(w,y)) - F(w,y)\|^2\} \\
&\quad + \gamma(w)[\|S(w,A(w,x)) - F(w,y)\|^2 + \|T(w,B(w,y)) - E(w,x)\|^2], \quad (1)
\end{aligned}$$

for $x, y \in H$ and $w \in \Omega$, where $\alpha, \beta, \gamma : \Omega \to [0, 1)$ are measurable mappings such that for all $w \in \Omega$,

$$2\alpha(w) + \beta(w) + 4\gamma(w) < 1. \quad (2)$$

3 Main Results

In this section, we prove a common random fixed point theorem for six weakly compatible random operators in separable Hilbert spaces without using the continuity of these mappings.

Theorem 3.1 *Let C be a nonempty closed subset of a separable Hilbert space H. Let $E, F, S, , T, A$ and $B : \Omega \times C \to C$ be six random mappings defined on C such that for $w \in \Omega$, E, F, S, T, A and $B : \Omega \times C \to C$ satisfy condition* ($\mathbf{A}$) *and the following conditions:*

$$E(w, H) \subseteq T(w, B(w, H)) \ , \ F(w, H) \subseteq S(w, A(w, H)). \tag{3}$$

$$EA = AE, \ \ SA = AS, \ \ BF = FB, \ \ TB = BT. \tag{4}$$

$$\text{The pairs } (E, SA) \text{ and } (F, TB) \text{ are weakly compatible.} \tag{5}$$

Then $E, F, S, , T, A$ and B have a unique common random fixed point.

Proof. Let the function $g_0 : \Omega \to C$ be an arbitrary measurable function on Ω. By (3) there exists a function $g_1 : \Omega \to C$ such that for $w \in \Omega$, $T(w, B(w, g_1(w))) = E(w, g_0(w))$ and for this function $g_1 : \Omega \to C$ we can choose another function $g_2 : \Omega \to C$ such that for $w \in \Omega$, $F(w, g_1(w)) = S(w, A(w, g_2(w)))$ and so on. By using the method of induction we can define a sequence of functions $y_n(w)$, $w \in \Omega$ as following:

$$\begin{aligned} y_{2n}(w) &= T(w, B(w, g_{2n+1}(w))) = E(w, g_{2n}(w)), \\ y_{2n+1}(w) &= S(w, A(w, g_{2n+2}(w))) = F(w, g_{2n+1}(w)), w \in \Omega, n = 0, 1, 2, . . \end{aligned} \tag{6}$$

From (1) we have for $w \in \Omega$ that

$$\begin{aligned} \|y_{2n}(w) - y_{2n+1}(w)\|^2 &= \|E(w, g_{2n}(w)) - F(w, g_{2n+1}(w))\|^2 \\ &\leq \alpha(w) \max\{\|S(w,A(w,g_{2n}(w)))-T(w,B(w,g_{2n+1}(w)))\|^2 \\ &, \ \|S(w,A(w,g_{2n}(w)))-E(w,g_{2n}(w))\|^2 \\ &, \ \|T(w,B(w,g_{2n+1}(w)))-F(w,g_{2n+1}(w))\|^2 \\ &, \ \tfrac{\|S(w,A(w,g_{2n}(w)))-F(w,g_{2n+1}(w))\|^2+\|T(w,B(w,g_{2n+1}(w)))-E(w,g_{2n}(w))\|^2}{2}\} \\ &+ \beta(w) \max\{\|S(w,A(w,g_{2n}(w)))-E(w,g_{2n}(w))\|^2 \\ &, \ \|T(w,B(w,g_{2n+1}(w)))-F(w,g_{2n+1}(w))\|^2\} \\ &+ \gamma(w)[\|S(w,A(w,g_{2n}(w)))-F(w,g_{2n+1}(w))\|^2 \\ &+ \|T(w,B(w,g_{2n+1}(w)))-E(w,g_{2n}(w))\|^2]. \end{aligned}$$

It follows by (6) that

$$\|y_{2n}(w) - y_{2n+1}(w)\|^2 \leq \alpha(w) \max\{\|y_{2n-1}(w) - y_{2n}(w)\|^2$$

$$
\begin{aligned}
&, \quad \|y_{2n-1}(w) - y_{2n}(w)\|^2, \|y_{2n}(w) - y_{2n+1}(w)\|^2 \\
&, \quad \frac{\|y_{2n-1}(w) - y_{2n+1}(w)\|^2 + \|y_{2n}(w) - y_{2n}(w)\|^2}{2}\} \\
&+ \quad \beta(w) \max\{\|y_{2n-1}(w) - y_{2n}(w)\|^2 \\
&, \quad \|y_{2n}(w) - y_{2n+1}(w)\|^2\} \\
&+ \quad \gamma(w)[\|y_{2n-1}(w) - y_{2n+1}(w)\|^2 \\
&+ \quad \|y_{2n}(w) - y_{2n}(w)\|^2]. \qquad (7)
\end{aligned}
$$

By parallelogram law, $\|x+y\|^2 + \|x-y\|^2 \leq 2[\|x\|^2 + \|y\|^2]$ which implies that $\|x+y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$, we can write

$$
\begin{aligned}
\|y_{2n-1}(w) - y_{2n+1}(w)\|^2 &= \|y_{2n-1}(w) - y_{2n}(w) + y_{2n}(w) - y_{2n+1}(w)\|^2 \\
&\leq 2\|y_{2n-1}(w) - y_{2n}(w)\|^2 + 2\|y_{2n}(w) - y_{2n+1}(w)\|^2. \qquad (8)
\end{aligned}
$$

Applying (8) in (7) we get

$$
\begin{aligned}
\|y_{2n}(w) - y_{2n+1}(w)\|^2 &\leq \alpha(w) \max\{\|y_{2n-1}(w) - y_{2n}(w)\|^2 \\
&, \quad \|y_{2n-1}(w) - y_{2n}(w)\|^2, \|y_{2n}(w) - y_{2n+1}(w)\|^2 \\
&, \quad \frac{2\|y_{2n-1}(w) - y_{2n}(w)\|^2 + 2\|y_{2n}(w) - y_{2n+1}(w)\|^2}{2}\} \\
&+ \quad \beta(w) \max\{\|y_{2n-1}(w) - y_{2n}(w)\|^2 \\
&, \quad \|y_{2n}(w) - y_{2n+1}(w)\|^2\} \\
&+ \quad \gamma(w)[2\|y_{2n-1}(w) - y_{2n}(w)\|^2 \\
&+ \quad 2\|y_{2n}(w) - y_{2n+1}(w)\|^2]. \qquad (9)
\end{aligned}
$$

If $\|y_{2n}(w) - y_{2n+1}(w)\|^2 > \|y_{2n-1}(w) - y_{2n}(w)\|^2$, then by (9) and (2) we have

$$
\begin{aligned}
\|y_{2n}(w) - y_{2n+1}(w)\|^2 &< 2\alpha(w)\|y_{2n}(w) - y_{2n+1}(w)\|^2 + \beta(w)\|y_{2n}(w) - y_{2n+1}(w)\|^2 \\
&+ 4\gamma(w)\|y_{2n}(w) - y_{2n+1}(w)\|^2 \\
&= (2\alpha(w) + \beta(w) + 4\gamma(w))\|y_{2n}(w) - y_{2n+1}(w)\|^2 \\
&< \|y_{2n}(w) - y_{2n+1}(w)\|^2.
\end{aligned}
$$

A contradiction.
It follows that $\|y_{2n}(w) - y_{2n+1}(w)\|^2 \leq \|y_{2n-1}(w) - y_{2n}(w)\|^2$.
Applying this in (9), we obtain

$$\|y_{2n}(w) - y_{2n+1}(w)\|^2 \leq (2\alpha(w) + \beta(w) + 4\gamma(w))\|y_{2n-1}(w) - y_{2n}(w)\|^2.$$

Hence

$$\|y_{2n}(w) - y_{2n+1}(w)\| \leq (2\alpha(w) + \beta(w) + 4\gamma(w))^{\frac{1}{2}}\|y_{2n-1}(w) - y_{2n}(w)\|.$$

By (2) we have $k = (2\alpha(w) + \beta(w) + 4\gamma(w))^{\frac{1}{2}} < 1$.
In general,

$$\|y_n(w) - y_{n+1}(w)\| \leq k\|y_{n-1}(w) - y_n(w)\|,$$

which implies that

$$\|y_n(w) - y_{n+1}(w)\| \leq k^n\|y_0(w) - y_1(w)\|, w \in \Omega.$$

Now, we will prove that for $w \in \Omega$, $\{y_n(w)\}$ is a Cauchy sequence in C.
For positive integer p we have

$$\begin{aligned}
\|y_n(w) - y_{n+p}(w)\| &= \|y_n(w) - y_{n+1}(w) + y_{n+1}(w) \\
&+ ... + y_{n+p-1}(w) - y_{n+p}(w)\| \\
&\leq \|y_n(w) - y_{n+1}(w)\| + \|y_{n+1}(w) - y_{n+2}(w)\| \\
&+ ... + \|y_{n+p-1}(w) - y_{n+p}(w)\| \\
&\leq [k^n + k^{n+1} + ... + k^{n+p-1}]\|y_0(w) - y_1(w)\| \\
&= k^n[1 + k + k^2 + ... + k^{p-1}]\|y_0(w) - y_1(w)\| \\
&\leq \frac{k^n}{1-k}\|y_0(w) - y_1(w)\| \\
&\to 0 \ (as\ n \to \infty), w \in \Omega.
\end{aligned}$$

It follows that $\{y_n(w)\}$ is a Cauchy sequence and hence is convergent in the closed subset C of H. So that, $\{y_n(w)\} \to \{y(w)\}$ as $n \to \infty$ for $w \in \Omega$. Since C is closed, $\{y(w)\}$ is a function from C to C and consequently the subsequences $\{E(w, g_{2n}(w))\}$, $\{F(w, g_{2n+1}(w))\}$, $\{T(w, B(w, g_{2n+1}(w)))\}$ and $\{S(w, A(w, g_{2n+2}(w)))\}$ of $\{y_n(w)\}$ also converge to $\{y(w)\}$.
Now, since $E(w, H) \subseteq T(w, B(w, H))$, there exists $h(w) \in C$ such that

$$y(w) = T(w, B(w, h(w))) \quad for \quad w \in \Omega. \tag{10}$$

Using (1) we obtain

$$\begin{aligned}
\|E(w,g_{2n}(w))-F(w,h(w))\|^2 &\leq \alpha(w)\max\{\|S(w,A(w,g_{2n}(w)))-T(w,B(w,h(w)))\|^2 \\
&, \|S(w,A(w,g_{2n}(w)))-E(w,g_{2n}(w))\|^2 \\
&, \|T(w,B(w,h(w)))-F(w,h(w))\|^2 \\
&, \frac{\|S(w,A(w,g_{2n}(w)))-F(w,h(w))\|^2+\|T(w,B(w,h(w)))-E(w,g_{2n}(w))\|^2}{2}\} \\
&+ \beta(w)\max\{\|S(w,A(w,g_{2n}(w)))-E(w,g_{2n}(w))\|^2 \\
&, \|T(w,B(w,h(w)))-F(w,h(w))\|^2\} \\
&+ \gamma(w)[\|S(w,A(w,g_{2n}(w)))-F(w,h(w))\|^2 \\
&+ \|T(w,B(w,h(w)))-E(w,g_{2n}(w))\|^2].
\end{aligned}$$

Taking the limit on both sides of the above inequality as $n \to \infty$, and using (10) we obtain

$$\begin{aligned}
\|y(w) - F(w, h(w))\|^2 &\leq \alpha(w) \max\{\|y(w) - y(w)\|^2, \|y(w) - y(w)\|^2 \\
&, \|y(w)) - F(w, h(w))\|^2 \\
&, \frac{\|y(w) - F(w, h(w))\|^2 + \|y(w) - y(w)\|^2}{2}\} \\
&+ \beta(w) \max\{\|y(w) - y(w)\|^2, \|y(w) - F(w, h(w))\|^2\} \\
&+ \gamma(w)[\|y(w) - F(w, h(w))\|^2 + \|y(w) - y(w)\|^2].
\end{aligned}$$

It follows that

$$\|y(w) - F(w, h(w))\|^2 \leq (\alpha(w) + \beta(w) + \gamma(w))\|y(w) - F(w, h(w))\|^2,$$

which leads to the following

$$y(w) = F(w, h(w)) \quad for \quad w \in \Omega. \tag{11}$$

From (10) and (11), we have $F(w, h(w)) = T(w, B(w, h(w)))$.
Since $\{F, TB\}$ are weakly compatible, then they commute at their coincidence point $h(w)$, i.e.

$$\begin{aligned}
F(w, T(w, B(w, h(w)))) &= T(w, B(w, F(w, h(w)))) \\
\Rightarrow F(w, y(w)) &= T(w, B(w, y(w))) \qquad (12)
\end{aligned}$$

Similarly, since $F(w, H) \subseteq S(w, A(w, H))$, there exists $f(w) \in C$ such that

$$y(w) = S(w, A(w, f(w))) \quad for \quad w \in \Omega. \tag{13}$$

Again using (1), we have

$$\begin{aligned}
\|E(w,f(w)) - F(w,g_{2n+1}(w))\|^2 &\leq \alpha(w) \max\{\|S(w,A(w,f(w))) - T(w,B(w,g_{2n+1}(w)))\|^2 \\
&, \|S(w,A(w,f(w))) - E(w,f(w))\|^2 \\
&, \|T(w,B(w,g_{2n+1}(w))) - F(w,g_{2n+1}(w))\|^2 \\
&, \frac{\|S(w,A(w,f(w))) - F(w,g_{2n+1}(w))\|^2 + \|T(w,B(w,g_{2n+1}(w))) - E(w,f(w))\|^2}{2}\} \\
&+ \beta(w) \max\{\|S(w,A(w,f(w))) - E(w,f(w))\|^2 \\
&, \|T(w,B(w,g_{2n+1}(w))) - F(w,g_{2n+1}(w))\|^2\} \\
&+ \gamma(w)[\|S(w,A(w,f(w))) - F(w,g_{2n+1}(w))\|^2 \\
&+ \|T(w,B(w,g_{2n+1}(w))) - E(w,f(w))\|^2].
\end{aligned}$$

Taking the limit on both sides of the above inequality as $n \to \infty$, and using (13) we obtain

$$\|E(w, f(w)) - y(w)\|^2 \leq \alpha(w) \max\{\|y(w) - y(w)\|^2, \|y(w) - E(w, f(w))\|^2$$

$$
\begin{aligned}
&, \quad \|y(w)-y(w)\|^2 \\
&, \quad \frac{\|y(w)-y(w)\|^2+\|y(w)-E(w,f(w))\|^2}{2}\} \\
&+ \quad \beta(w)\max\{\|y(w)-E(w,f(w))\|^2,\|y(w)-y(w)\|^2\} \\
&+ \quad \gamma(w)[\|y(w)-y(w)\|^2+\|y(w)-E(w,f(w))\|^2].
\end{aligned}
$$

It follows that

$$\|E(w,f(w))-y(w)\|^2 \le (\alpha(w)+\beta(w)+\gamma(w))\|E(w,f(w))-y(w)\|^2.$$

Hence

$$E(w,f(w)) = y(w) \quad for \quad w\in\Omega. \tag{14}$$

Using(13) and (14), we have $S(w,A(w,f(w))) = E(w,f(w))$.
Since $\{E,SA\}$ are weakly compatible, then they commute at their coincidence point $f(w)$, i.e.

$$
\begin{aligned}
S(w,A(w,E(w,f(w)))) &= E(w,S(w,A(w,f(w)))) \\
\Rightarrow S(w,A(w,y(w))) &= E(w,y(w)). \qquad (15)
\end{aligned}
$$

Now, we show the existence of a random fixed point. Consider for $w\in\Omega$, and by parallelogram law we have that,

$$
\begin{aligned}
\|E(w,y(w))-y(w)\|^2 &= \|E(w,y(w))-y_{2n+1}(w)+y_{2n+1}(w)-y(w)\|^2 \\
&\le 2\|E(w,y(w))-y_{2n+1}(w)\|^2+2\|y_{2n+1}(w)-y(w)\|^2 \\
&= 2\|E(w,y(w))-F(w,g_{2n+1}(w))\|^2+2\|y_{2n+1}(w)-y(w)\|^2 \\
&\le 2\alpha(w)\max\{\|S(w,A(w,y(w)))-T(w,B(w,g_{2n+1}(w)))\|^2 \\
&, \quad \|S(w,A(w,y(w)))-E(w,y(w))\|^2,\|T(w,B(w,g_{2n+1}(w)))-F(w,g_{2n+1}(w))\|^2 \\
&, \quad \frac{\|S(w,A(w,y(w)))-F(w,g_{2n+1}(w))\|^2+\|T(w,B(w,g_{2n+1}(w)))-E(w,y(w))\|^2}{2}\} \\
&+ 2\beta(w)\max\{\|S(w,A(w,y(w)))-E(w,y(w))\|^2 \\
&, \quad \|T(w,B(w,g_{2n+1}(w)))-F(w,g_{2n+1}(w))\|^2\} \\
&+ 2\gamma(w)[\|S(w,A(w,y(w)))-F(w,g_{2n+1}(w))\|^2 \\
&+ \|T(w,B(w,g_{2n+1}(w)))-E(w,y(w))\|^2]+2\|y_{2n+1}(w)-y(w)\|^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|E(w,y(w))-y(w)\|^2 &\le 2\alpha(w)\max\{\|S(w,A(w,y(w)))-y(w)\|^2,\|S(w,A(w,y(w)))-E(w,y(w))\|^2 \\
&, \quad \|y(w)-y(w)\|^2, \frac{\|S(w,A(w,y(w)))-y(w)\|^2+\|y(w)-E(w,y(w))\|^2}{2}\} \\
&+ 2\beta(w)\max\{\|S(w,A(w,y(w)))-E(w,y(w))\|^2,\|y(w)-y(w)\|^2\} \\
&+ 2\gamma(w)[\|S(w,A(w,y(w)))-y(w)\|^2+\|y(w)-E(w,y(w))\|^2] \\
&+ 2\|y(w)-y(w)\|^2.
\end{aligned}
$$

Using (15) we obtain

$$\begin{aligned}\|E(w,y(w))-y(w)\|^2 &\leq (2\alpha(w)+4\gamma(w))\|E(w,y(w))-y(w)\|^2\\ &< \|E(w,y(w))-y(w)\|^2.\end{aligned}$$

It follows that

$$E(w,y(w)) = y(w). \tag{16}$$

From (15) and (16) we have

$$E(w,y(w)) = S(w,A(w,y(w))) = y(w) \quad for \quad w\in\Omega. \tag{17}$$

Similarly, we can show that

$$F(w,y(w)) = T(w,B(w,y(w))) = y(w) \quad for \quad w\in\Omega. \tag{18}$$

It follows from the construction of $\{y_n(w)\}$ for $w\in\Omega$ that $\{y_n(w)\}$ is a sequence of measurable functions and since $y(w)$ is a pointwise limit of a measurable sequence $\{y_n(w)\}$, it follows that $y(w)$ is also measurable function and by (17) and(18), $y(w):\Omega\to C$ is a common random fixed point of E, F, SA and TB.
Next we prove $y(w) = S(w,y(w)) = A(w,y(w)) = T(w,y(w)) = B(w,y(w))$. Since $AE = EA$ and using (1) we have

$$\begin{aligned}\|A(w,y(w))-y(w)\|^2 &= \|A(w,E(w,y(w)))-F(w,y(w))\|^2=\|E(w,A(w,y(w)))-F(w,y(w))\|^2\\ &\leq \alpha(w)\max\{\|S(w,A(w,A(w,y(w))))-T(w,B(w,y(w)))\|^2\\ &, \|S(w,A(w,A(w,y(w))))-E(w,A(w,y(w)))\|^2,\|T(w,B(w,y(w)))-F(w,y(w))\|^2\\ &, \frac{\|S(w,A(w,A(w,y(w))))-F(w,y(w))\|^2+\|T(w,B(w,y(w)))-E(w,A(w,y(w)))\|^2}{2}\}\\ &+ \beta(w)\max\{\|S(w,A(w,A(w,y(w))))-E(w,A(w,y(w)))\|^2\\ &, \|T(w,B(w,y(w)))-F(w,y(w))\|^2\}\\ &+ \gamma(w)[\|S(w,A(w,A(w,y(w))))-F(w,y(w))\|^2\\ &+ \|T(w,B(w,y(w)))-E(w,A(w,y(w)))\|^2].\end{aligned} \tag{19}$$

Since $AE = EA$ and $SA = AS$ we have $E(w,A(w,y(w))) = A(w,E(w,y(w))) = A(w,y(w))$ and $S(w,A(w,A(w,y(w)))) = A(w,S(w,A(w,y(w)))) = A(w,y(w))$. Applying this in (19) we obtain

$$\begin{aligned}\|A(w,y(w))-y(w)\|^2 &\leq \alpha(w)\max\{\|A(w,y(w))-y(w)\|^2\\ &, \|A(w,y(w))-A(w,y(w))\|^2, \|y(w)-y(w)\|^2\\ &, \frac{\|A(w,y(w))-y(w)\|^2+\|y(w)-A(w,y(w))\|^2}{2}\}\\ &+ \beta(w)\max\{\|A(w,y(w))-A(w,y(w))\|^2\\ &, \|y(w)-y(w)\|^2\}\\ &+ \gamma(w)[\|A(w,y(w))-y(w)\|^2+\|y(w)-A(w,y(w))\|^2].\end{aligned}$$

It follows that

$$\begin{aligned}\|A(w,y(w))-y(w)\|^2 &\leq (\alpha(w)+2\gamma(w))\|A(w,y(w))-y(w)\|^2\\ &\Rightarrow A(w,y(w))=y(w). \qquad (20)\end{aligned}$$

Since $S(w,A(w,y(w)))=y(w)$ and by (20) we have $S(w,y(w))=y(w)$, i.e. $S(w,y(w))=A(w,y(w))=y(w)$.
Again, since $BF=FB$ and using (1) we have

$$\begin{aligned}\|y(w)-B(w,y(w))\|^2 &= \|E(w,y(w))-B(w,F(w,y(w)))\|^2=\|E(w,y(w))-F(w,B(w,y(w)))\|^2\\ &\leq \alpha(w)\max\{\|S(w,A(w,y(w)))-T(w,B(w,B(w,y(w))))\|^2\\ &, \|S(w,A(w,y(w)))-E(w,y(w))\|^2,\|T(w,B(w,B(w,y(w))))-F(w,B(w,y(w)))\|^2\\ &, \frac{\|S(w,A(w,y(w)))-F(w,B(w,y(w)))\|^2+\|T(w,B(w,B(w,y(w))))-E(w,y(w))\|^2}{2}\}\\ &+ \beta(w)\max\{\|S(w,A(w,y(w)))-E(w,y(w))\|^2\\ &, \|T(w,B(w,B(w,y(w))))-F(w,B(w,y(w)))\|^2\}\\ &+ \gamma(w)[\|S(w,A(w,y(w)))-F(w,B(w,y(w)))\|^2\\ &+ \|T(w,B(w,B(w,y(w))))-E(w,y(w))\|^2]\\ &. \qquad (21)\end{aligned}$$

Since $FB=BF$ and $TB=BT$ we have $F(w,B(w,y(w)))=B(w,F(w,y(w)))=B(w,y(w))$ and $T(w,B(w,B(w,y(w))))=B(w,T(w,B(w,y(w))))=B(w,y(w))$. Applying this in (21) we get

$$\begin{aligned}\|y(w)-B(w,y(w))\|^2 &\leq \alpha(w)\max\{\|y(w)-B(w,y(w))\|^2\\ &, \|y(w)-y(w)\|^2, \|B(w,y(w))-B(w,y(w))\|^2\\ &, \frac{\|y(w)-B(w,y(w))\|^2+\|B(w,y(w))-y(w)\|^2}{2}\}\\ &+ \beta(w)\max\{\|y(w)-y(w)\|^2\\ &, \|B(w,y(w))-B(w,y(w))\|^2\}\\ &+ \gamma(w)[\|y(w)-B(w,y(w))\|^2+\|B(w,y(w))-y(w)\|^2].\end{aligned}$$

It follows that

$$\begin{aligned}\|y(w)-B(w,y(w))\|^2 &\leq (\alpha(w)+2\gamma(w))\|B(w,y(w))-y(w)\|^2\\ &\Rightarrow B(w,y(w))=y(w). \qquad (22)\end{aligned}$$

Since $T(w,B(w,y(w)))=y(w)$ and by (22) we have $T(w,y(w))=y(w)$, i.e. $T(w,y(w))=B(w,y(w))=y(w)$.

Finally, for the uniqueness of the common random fixed point $y(w)$ of E,F,S,T,A and B , let $p(w):\Omega\to C$ be another common random fixed point of E, F, S, T, A and B, using (1) we obtain

$$\begin{aligned}\|y(w)-p(w)\|^2 &= \|E(w,y(w))-F(w,p(w))\|^2\\ &\leq \alpha(w)\max\{\|S(w,A(w,y(w)))-T(w,B(w,p(w)))\|^2\\ &, \|S(w,A(w,y(w)))-E(w,y(w))\|^2, \|T(w,B(w,p(w)))-F(w,p(w))\|^2\\ &, \frac{\|S(w,A(w,y(w)))-F(w,p(w))\|^2+\|T(w,B(w,p(w)))-E(w,y(w))\|^2}{2}\}\\ &+ \beta(w)\max\{\|S(w,A(w,y(w)))-E(w,y(w))\|^2, \|T(w,B(w,p(w)))-F(w,p(w))\|^2\}\\ &+ \gamma(w)[\|S(w,A(w,y(w)))-F(w,p(w))\|^2+\|T(w,B(w,p(w)))-E(w,y(w))\|^2],\end{aligned}$$

which yields

$$\begin{aligned}\|y(w)-p(w)\|^2 &\leq \alpha(w)\max\{\|y(w)-p(w)\|^2, \|y(w)-y(w)\|^2\\ &, \|p(w)-p(w)\|^2, \frac{\|y(w)-p(w)\|^2+\|p(w)-y(w)\|^2}{2}\}\\ &+ \beta(w)\max\{\|y(w)-y(w)\|^2, \|p(w)-p(w)\|^2\}\\ &+ \gamma(w)[\|y(w)-p(w)\|^2+\|p(w)-y(w)\|^2].\end{aligned}$$

This implies

$$\begin{aligned}\|y(w)-p(w)\|^2 &\leq (\alpha(w)+2\gamma(w))\|y(w)-p(w)\|^2\\ &< \|y(w)-p(w)\|^2\\ \Rightarrow y(w) = p(w) &\quad for \quad w\in\Omega.\end{aligned}$$

The proof of the theorem is completed.
If we put $A = B = I$ (where I is the identity mapping on H) in Theorem 3.1, we obtain the following result:

Corollary 3.2 *Let C be a nonempty closed subset of a separable Hilbert space H. Let E, F, S and $T : \Omega\times C \to C$ be four random mappings satisfing the following conditions:*

$$E(w,H)\subseteq T(w,H),\quad F(w,H)\subseteq S(w,H). \tag{23}$$

$$\textit{The pairs } (E,S) \textit{ and } (F,T) \textit{ are weakly compatible.} \tag{24}$$

$$\begin{aligned}\|E(w,x)-F(w,y)\|^2 &\leq \alpha(w)\max\{\|S(w,x)-T(w,y)\|^2, \|S(w,x)-E(w,x)\|^2\\ &, \|T(w,y)-F(w,y)\|^2\\ &, \frac{\|S(w,x)-F(w,y)\|^2+\|T(w,y)-E(w,x)\|^2}{2}\}\\ &+ \beta(w)\max\{\|S(w,x)-E(w,x)\|^2, \|T(w,y)-F(w,y)\|^2\}\\ &+ \gamma(w)[\|S(w,x)-F(w,y)\|^2+\|T(w,y)-E(w,x)\|^2],\end{aligned} \tag{25}$$

for $x, y \in H$ and $w \in \Omega$, where $\alpha, \beta, \gamma : \Omega \to [0, 1)$ are measurable mappings such that for all $w \in \Omega$,

$$2\alpha(w) + \beta(w) + 4\gamma(w) < 1.$$

Then E, F, S and T have a unique common random fixed point.

If $S = T$ and $E = F$ in corollary 3.2, we have the following result:

Corollary 3.3 *Let C be a nonempty closed subset of a separable Hilbert space H. Let E and $S : \Omega \times C \to C$ be two random mappings satisfing the following conditions:*

$$E(w, H) \subseteq S(w, H). \tag{26}$$

$$\textit{The pair } (E, S) \textit{ is weakly compatible.} \tag{27}$$

$$\begin{aligned} \|E(w,x) - E(w,y)\|^2 \leq\ & \alpha(w) \max\{\|S(w,x)-S(w,y)\|^2, \|S(w,x)-E(w,x)\|^2 \\ , \ & \|S(w,y)-E(w,y)\|^2 \\ , \ & \tfrac{\|S(w,x)-E(w,y)\|^2 + \|S(w,y)-E(w,x)\|^2}{2}\} \\ + \ & \beta(w) \max\{\|S(w,x)-E(w,x)\|^2, \|S(w,y)-E(w,y)\|^2\} \\ + \ & \gamma(w)[\|S(w,x)-E(w,y)\|^2 + \|S(w,y)-E(w,x)\|^2], \end{aligned} \tag{28}$$

for $x, y \in H$ and $w \in \Omega$, where $\alpha, \beta, \gamma : \Omega \to [0, 1)$ are measurable mappings such that for all $w \in \Omega$,

$$2\alpha(w) + \beta(w) + 4\gamma(w) < 1.$$

Then E and S have a unique common random fixed point.

4 Open Problems

(1) Is Theorem 3.1 true in a Polish metric space?
(2) Is Theorem 3.1 can be extended to more general contraction mappings?

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