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Quintic C^1-Spline Collocation Methods for Stiff Delay Differential Equations

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Abstract

In this paper, a new difference scheme based on C^1-quintic splines is derived for the numerical solution of the stiff delay differential equations. Convergence results shows that the methods have a convergence of order five. Moreover, the stability analysis properties of these methods have been studied. Finally, numerical results illustrating the behavior of the methods when faced with some difficult problems are presented.

Keywords: *Stiff delay differential equations, Quintic spline, Collocation methods, Stability analysis.*

1 Introduction

Delay differential equations (DDEs) are used to model a large variety of practical phenomena in the biosciences, engineering and control theory, and in many other areas of science and technology, in which the time evolution depends not only on present states but also on states at or near a given time in the past (see, e.g., [1, 10]). If we restrict the class of DDEs to a class in which the highest derivative is multiplied by a small parameter, then it is said to be a stiff delay differential equation (SDDEs). Such problems arise in the mathematical modeling of various practical phenomena, for example, in population dynamics [10], the study of bistable devices [2], description of the human pupil-light reflex [13], and variational problems in control theory [14].

In this paper we will be concerned with the numerical solution of the SDDEs:

$$\begin{aligned} &\varepsilon y'(x) = f(x, y(x), y(\alpha(x))), \quad x \in [a, b], \\ &y(x) = g(x) \ for \ \widetilde{a} \leq x < a, \end{aligned} \tag{1}$$

where $f \in C^5([a, b] \times R \times R)$ is Lipschitz continuous with respect to y, $\widetilde{a} = inf[\alpha(x)]$, $0 < \varepsilon \leq 1$ is the a positive small parameter. The function $\alpha(x) \leq x, x \in [a, b]$ is usually called the delay function, $g(x)$ is the initial function.

Several numerical methods that were originally designed for solving SDDEs, for example, El-Gendi [3] considered Chebyshev series for the numerical solution of DDEs, with single delay. El-Gendi methods for solving SDDEs were presented in [8], θ-methods for solving these equations were considered in [7, 11, 12]. Spline collocation methods for solving delay and neutral delay differential equations were considered by several authors (see, e.g., [4, 5, 6, 9]). Quintic C^2-spline integrating method for solving second-order ordinary initial value problems were studied in [15]. More detailed analysis for both the convergence and absolute stability was also given.

The outline of this paper is as follows: Section 2 contains an investigation of the existence, uniqueness, and the precise definition of spline collocation methods. Convergence results are given in Section 3 and it turned out that the method is fifth order. Section 4 is devoted to the stability analysis. In Section 5 three numerical results for both SDDEs and the system of SDDEs cases are given to illustrate the efficiency of our method. The last section is conclusion.

2 Description of the methods

Consider the initial value problem for the SDDEs (1). The basic idea is to generate a quintic spline $s \in C^1[a, b]$ which satisfies Eq. (1) at the interior knots $x_{i-3/4}, x_{i-1/2}, x_{i-1/4}$ as well as at x_i, where $I_i = [x_{i-1}, x_i]$ with $x_i = a + ih, i = 1(1)n, n = (b - a)/h, h$ is the stepsize. Let $S^{(1)}_{n,5} = \{s(x) : s \in C^1[a, b], s \in \Pi_5,$ for $x \in I_i, i = 1(1)n\}$, where Π_5 denotes the collection of all polynomials of degree ≤ 5. Using the notations

$$\begin{aligned} &s'_{i-1} = s'(x_{i-1}), \quad s'_{i-3/4} = s'(x_{i-3/4}), \quad s'_{i-1/2} = s'(x_{i-1/2}), \\ &s'_{i-1/4} = s'(x_{i-1/4}), \quad s'_i = s'(x_i), \ i = 1(1)n, \end{aligned}$$

a quintic spline functions $s \in S^{(1)}_{n,5}$ can be represented on each I_i by

$$s(x) = s_{i-1} + hA(t)s'_{i-1} + hB(t)s'_{i-3/4} + hC(t)s'_{i-1/2} + hD(t)s'_{i-1/4} + hE(t)s'_i, \tag{2a}$$

with

$$\begin{aligned}
&A(t) = t - \frac{25}{6}t^2 + \frac{70}{9}t^3 - \frac{20}{3}t^4 + \frac{32}{15}t^5,\ B(t) = 8t^2 - \frac{208}{9}t^3 + 24t^4 - \frac{128}{15}t^5,\\
&C(t) = -6t^2 + \frac{76}{3}t^3 - 32t^4 + \frac{192}{15}t^5, \quad D(t) = \frac{8}{3}t^2 - \frac{112}{9}t^3 + \frac{56}{3}t^4 - \frac{128}{15}t^5,\\
&E(t) = -\frac{1}{2}t^2 + \frac{22}{9}t^3 - 4t^4 + \frac{32}{15}t^5,
\end{aligned} \tag{2b}$$

and $x = x_{i-1} + th,\ t \in [0,1]$. Since $s \in S^{(1)}_{n,5}$, we have

$$s_i = s_{i-1} + \frac{7}{90}hs'_{i-1} + \frac{16}{45}hs'_{i-3/4} + \frac{2}{15}hs'_{i-1/2} + \frac{16}{45}hs'_{i-1/4} + \frac{7}{90}hs'_i, i = 1(1)n, \tag{3a}$$

and hence s is uniquely determined in $[a,b]$. Scheme (2a) can be exploited to generate successively a quintic spline $s \in C^1[a,b]$ to find an approximation to the exact solution of Eq. (1). Since f satisfy the Lipschitz condition in $([a,b] \times R \times R)$ then the approximate spline solution $s(x)$ to the exact solution $y(x)$ of Eq. (1) will be constructed as follows: for $i = 1(1)n$

$$\begin{aligned}
s_{i-3/4} &= s_{i-1} + \frac{251}{2880}hs'_{i-1} + \frac{323}{1440}hs'_{i-3/4} - \frac{11}{120}hs'_{i-1/2} + \frac{53}{1440}hs'_{i-1/4} - \frac{19}{2880}hs'_i,\\
s_{i-1/2} &= s_{i-1} + \frac{29}{360}hs'_{i-1} + \frac{31}{90}hs'_{i-3/4} + \frac{1}{15}hs'_{i-1/2} + \frac{1}{90}hs'_{i-1/4} - \frac{1}{360}hs'_i,\\
s_{i-1/4} &= s_{i-1} + \frac{27}{320}hs'_{i-1} + \frac{51}{160}hs'_{i-3/4} + \frac{9}{40}hs'_{i-1/2} + \frac{21}{160}hs'_{i-1/4} - \frac{3}{320}hs'_i.
\end{aligned} \tag{3b}$$

It is easy to observe that $s(\alpha(x_j)) = g(\alpha(x_j)), j = i-3/4, i-1/2, i-1/4, i,$ when $\alpha(x_j) \leq a$, and if $\alpha(x_j) \in [x_{k-1}, x_k], k = 1(1)i$ then $s(\alpha(x_j))$ can be calculated from Eq. (2):

$$\begin{aligned}
s(\alpha(x_j)) =& s_{k-1} + hA(\xi)s'_{k-1} + hB(\xi)s'_{k-3/4} + hC(\xi)s'_{k-1/2}\\
&+ hD(\xi)s'_{k-1/4} + hE(\xi)s'_k,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
&A(\xi) = \xi - \frac{25}{6}\xi^2 + \frac{70}{9}\xi^3 - \frac{20}{3}\xi^4 + \frac{32}{15}\xi^5,\ B(\xi) = 8\xi^2 - \frac{208}{9}\xi^3 + 24\xi^4 - \frac{128}{15}\xi^5,\\
&C(\xi) = -6\xi^2 + \frac{76}{3}\xi^3 - 32\xi^4 + \frac{192}{15}\xi^5, \quad D(\xi) = \frac{8}{3}\xi^2 - \frac{112}{9}\xi^3 + \frac{56}{3}\xi^4 - \frac{128}{15}\xi^5,\\
&E(\xi) = -\frac{1}{2}\xi^2 + \frac{22}{9}\xi^3 - 4\xi^4 + \frac{32}{15}\xi^5, \qquad \xi = \frac{\alpha(x_j) - x_{k-1}}{h} \in [0,1].
\end{aligned}$$

We can write Eq. (1) as follow:

$$\varepsilon s'_j = f(x_j, s(x_j), s(\alpha(x_j))), \quad j = i - 3/4, i - 1/2, i - 1/4, i, \tag{5}$$

in each subinterval $[x_{i-1}, x_i], i = 1(1)n$. From Eqs. (3) and (4), system (5) can be solved for $s'_{i-3/4}, s'_{i-1/2}, s'_{i-1/4}, s'_i$ by any numerical method.

Theorem 2.1. *If f satisfies Lipschitz condition, and if*

$$h < \frac{144}{179L}, \tag{6}$$

then there exists a unique spline approximation solution of Eq. (1) given by system (3).

Proof. It is sufficient to prove that $\underline{S}_i = (s_{i-3/4}, s_{i-1/2}, s_{i-1/4}, s_i)^T$ can be uniquely determined for an arbitrary given s_{i-1}. Since, we can write system (3) as follows:

$$\underline{S}_i = M_0 s_{i-1} + hM_1 f_{i-1} + hM_2 \underline{f}_i, \tag{7}$$

where

$$M_0 = (1, 1, 1, 1)^T, \quad M_1 = (\frac{251}{2880}, \frac{29}{360}, \frac{27}{320}, \frac{7}{90})^T,$$

$$M_2 = \begin{bmatrix} \frac{323}{1440} & -\frac{11}{120} & \frac{53}{1440} & -\frac{19}{2880} \\ \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & -\frac{1}{360} \\ \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & -\frac{3}{320} \\ \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \end{bmatrix},$$

$\underline{f}_i = (f_{i-3/4}, f_{i-1/2}, f_{i-1/4}, f_i)^T$, from Eq. (7), we have

$$\underline{S}_{i,1} = M_0 s_{i-1} + hM_1 f_{i-1,1} + hM_2 \underline{f}_{i,1},$$
$$\underline{S}_{i,2} = M_0 s_{i-1} + hM_1 f_{i-1,2} + hM_2 \underline{f}_{i,2}.$$

Thus, $\underline{S}_{i,1}$ and $\underline{S}_{i,2}$ can be written in the form

$$\underline{S}_{i,1} = \underline{Q}_{i,1}(s_{i-3/4,1}, s_{i-1/2,1}, s_{i-1/4,1}, s_{i,1}),$$
$$\underline{S}_{i,2} = \underline{Q}_{i,2}(s_{i-3/4,2}, s_{i-1/2,2}, s_{i-1/4,2}, s_{i,2}).$$

Applying $\|.\|_1$, Lipschitz condition, we get

$$\begin{aligned}
\|\underline{Q}_{i,1} - \underline{Q}_{i,2}\| &= \|(M_0 s_{i-1} + hM_1 f_{i-1,1} + hM_2 \underline{f}_{i,1}) \\
&\quad - (M_0 s_{i-1} + hM_1 f_{i-1,2} + hM_2 \underline{f}_{i,2})\| \\
&\leq \Big\{ \|M_1\| h |f_{i-1,1} - f_{i-1,2}| + \|M_2\| h \Big(|f_{i-3/4,1} - f_{i-3/4,2}| \\
&\quad + |f_{i-1/2,1} - f_{i-1/2,2}| + |f_{i-1/4,1} - f_{i-1/4,2}| + |f_{i,1} - f_{i,2}| \Big) \Big\} \\
&< \Big\{ \frac{95}{288} h L_1 |s_{i-1,1} - s_{i-1,2}| + \frac{179}{144} h \Big(L_2 |s_{i-3/4,1} - s_{i-3/4,2}| \\
&\quad + L_3 |s_{i-1/2,1} - s_{i-1/2,2}| + L_4 |s_{i-1/4,1} - s_{i-1/4,2}| + L_5 |s_{i,1} - s_{i,2}| \Big) \Big\} \\
&< \frac{179}{144} h L \Big\{ |s_{i-1,1} - s_{i-1,2}| + |s_{i-3/4,1} - s_{i-3/4,2}| \\
&\quad + |s_{i-1/2,1} - s_{i-1/2,2}| + |s_{i-1/4,1} - s_{i-1/4,2}| + |s_{i,1} - s_{i,2}| \Big\}
\end{aligned}$$

where

$$L = max(L_1, L_2, L_3, L_4, L_5).$$

Thus, the function $\underline{Q}_i$ defines a contraction mapping, if $(\frac{179}{144})hL < 1$, which satisfies Eq. (6). Hence there exists a unique $\underline{S}_i$ that satisfies

$$\underline{S}_i = \underline{Q}_i(s_{i-3/4}, s_{i-1/2}, s_{i-1/4}, s_i)$$

which may be found by iteration

$$\underline{S}_i^{p+1} = \underline{Q}_i(\underline{S}_i^p), \quad p = 0, 1, 2, ...$$

The proof of Theorem 2.1 is now complete.

3 Error analysis and order of convergence

In this section the emphasis is on conditions for convergence of the proposed method. It is shown that the method is a continuous extension of a multi-step method, and its derivative reproduces the values given by the well-known closed four-panel Newton-Cotes formula at the mesh points. A priori error estimates in L_∞-norm shows that the method is a fifth order as well as its first derivatives, according to the following:

Lemma 3.1. *Let $f \in C^6([a, b] \times R \times R)$, then*

$$E_i = O(h^5), \quad i = 0(1)n, \tag{8}$$

where $E_i = e_i + e_{\alpha_i}$,
$e_i = s_i - y_i$, with $y_i = y(x_i)$, and $e_{\alpha_i} = s_{\alpha_i} - y_{\alpha_i}$, with $y_{\alpha_i} = y(\alpha(x_i))$.

Proof. Since

$$s_i = s_{i-1} + \frac{h}{90}(7s'_{i-1} + 32s'_{i-3/4} + 12s'_{i-1/2} + 32s'_{i-1/4} + 7s'_i),$$

which is the well-known closed four-panel Newton-Cotes formula, applied to $y'(x)$, if $y \in C^6[a,b]$, then it follows that

$$e'_i = O(h^5).$$

But

$$e_i = e_{i-1} + \delta_i,$$

where

$$\delta_i = \frac{h}{90}(7e'_{i-1} + 32e'_{i-3/4} + 12e'_{i-1/2} + 32e'_{i-1/4} + 7e'_i) + O(h^6), \; e_0 = 0,$$

thus

$$e_i = \sum_{j=1}^{i} \delta_j$$

or

$$e_i = O(h^5). \tag{9}$$

Also, let $\alpha(x_i) \in [x_{k-1}, x_k], k \leq i$, then

$$\begin{aligned} s_{\alpha_i} =& s_{k-1} + h(A(\xi)s'_{k-1} + B(\xi)s'_{k-3/4} + C(\xi)s'_{k-1/2} \\ &+ D(\xi)s'_{k-1/4} + E(\xi)s'_k), \end{aligned}$$

$$e'_{\alpha_i} = O(h^5).$$

But

$$e_{\alpha_i} = e_{k-1} + \gamma_k,$$

where

$$\begin{aligned} \gamma_k =& h(A(\xi)e'_{k-1} + B(\xi)e'_{k-3/4} + C(\xi)e'_{k-1/2} + D(\xi)e'_{k-1/4} \\ &+ E(\xi)e'_k) + O(h^6), \; e_0 = 0, \end{aligned}$$

thus

$$e_{\alpha_i} = \sum_{j=1}^{k} \gamma_j$$

or

$$e_{\alpha_i} = O(h^5). \tag{10}$$

From Eqs. (9) and (10), we get $E_i = O(h^5)$. The proof of Lemma 3.1 is now completed.

We now turn to prove the following main theorem, which provides estimation for the global error for $s(x) - y(x)$ and its first derivative.

Theorem 3.1. *Let $f \in C^6([a, b] \times R \times R)$, then for all $x \in [a, b]$, we have*

$$|s^{(k)}(x) - y^{(k)}(x)| < C_k h^5, \;\; k = 0, 1, \tag{11}$$

where C_k denote generic constants independent of h, but dependent on the order of the various derivatives.

Proof. On $[x_{i-1}, x_i]$ we have

$$E_i'(x) = s'(x) - u'(x) + u'(x) - y'(x),$$

where $u'(x)$ is the quartic interpolant of $y'(x)$ at $x_{i-1}, x_{i-3/4}, x_{i-1/2}, x_{i-1/4}$ and x_i. It can be easily verified that

$$u'(x) = y'_{i-1}A'(t) + y'_{i-3/4}B'(t) + y'_{i-1/2}C'(t) + y'_{i-1/4}D'(t) + y'_i E'(t),$$

with $A'(t), ..., E'(t)$ be given from Eq. (2b).

But

$$s'(x) - u'(x) = E'_{i-1}A'(t) + E'_{i-3/4}B'(t) + E'_{i-1/2}C'(t) + E'_{i-1/4}D'(t) + E'_i E'(t).$$

Therefore

$$\begin{aligned} |s'(x) - u'(x)| &\le |E'_{i-1}||A'(t)| + |E'_{i-3/4}||B'(t)| + |E'_{i-1/2}||C'(t)| \\ &\quad + |E'_{i-1/4}||D'(t)| + |E'_i||E'(t)| \\ &\le |E'_{i-1}| + |E'_{i-3/4}| + |E'_{i-1/2}| + |E'_{i-1/4}| + |E'_i|, \end{aligned}$$

and using Lemma 3.1, it follows that

$$|s'(x) - u'(x)| = O(h^5).$$

Also from the construction of $u'(x)$, it follows that $|u'(x) - y'(x)| = O(h^5)$, provided $f \in C^6([a, b] \times R \times R)$. Hence, $|E'(x)| \le C_1 h^5$.

On $[x_{i-1}, x_i]$, we have

$$E(x) = \int_{x_{i-1}}^{x} E'(t)dt + E_{i-1},$$

or, using Lemma 3.1, we get

$$|E(x)| \le C_0 h^5.$$

This completes the proof of Theorem 3.1.

4 Stability analysis

Let us consider the following linear DDEs:

$$y'(x) = \lambda y(x) + qy(x - \tau) \tag{12}$$

as a stability test equation, where $\lambda, q \in C$ arbitrary, the delay τ is positive constant.

Definition 4.1. *A numerical method, applied to Eq. (12) is said to be P-stable if under the condition $Re(\lambda) < -|q|$, the numerical solution $s(x_i) \longrightarrow 0$ as $x_i \longrightarrow \infty$ for all h satisfying $\tilde{m}h = \tau,\ \tilde{m} \in N$. A region of P-stability is the set of all points $(h\lambda, hq)$ for which the method is P-stable.*

Applying the methods to Eq. (12)

$$\begin{aligned} &s'(x_j) = \lambda s(x_j) + qs(x_{j-m}),\ j = i - 3/4, i - 1/2, i - 1/4, i,\\ &i = 1(1)n,\ m \leq i, \end{aligned} \tag{13}$$

where $\tau = mh,\ s(x_{j-m}) = s(x_j - mh)$ and $x_{j-m} \in [x_{j-m-1}, x_{j-m}]$, we get from system (3):

$$\begin{aligned} s_{i-3/4} &= s_{i-1} + \frac{251}{2880}(zs_{i-1} + vs_{i-1-m}) + \frac{323}{1440}(zs_{i-3/4} + vs_{i-3/4-m})\\ &\quad -\frac{11}{120}(zs_{i-1/2} + vs_{i-1/2-m}) + \frac{53}{1440}(zs_{i-1/4} + vs_{i-1/4-m}) - \frac{19}{2880}(zs_i + vs_{i-m}),\\ s_{i-1/2} &= s_{i-1} + \frac{29}{360}(zs_{i-1} + vs_{i-1-m}) + \frac{31}{90}(zs_{i-3/4} + vs_{i-3/4-m})\\ &\quad +\frac{1}{15}(zs_{i-1/2} + vs_{i-1/2-m}) + \frac{1}{90}(zs_{i-1/4} + vs_{i-1/4-m}) - \frac{1}{360}(zs_i + vs_{i-m}),\\ s_{i-1/4} &= s_{i-1} + \frac{27}{320}(zs_{i-1} + vs_{i-1-m}) + \frac{51}{160}(zs_{i-3/4} + vs_{i-3/4-m})\\ &\quad +\frac{9}{40}(zs_{i-1/2} + vs_{i-1/2-m}) + \frac{21}{160}(zs_{i-1/4} + vs_{i-1/4-m}) - \frac{3}{320}(zs_i + vs_{i-m}),\\ s_i &= s_{i-1} + \frac{7}{90}(zs_{i-1} + vs_{i-1-m}) + \frac{16}{45}(zs_{i-3/4} + vs_{i-3/4-m})\\ &\quad +\frac{2}{15}(zs_{i-1/2} + vs_{i-1/2-m}) + \frac{16}{45}(zs_{i-1/4} + vs_{i-1/4-m}) + \frac{7}{90}(zs_i + vs_{i-m}), \end{aligned} \tag{14}$$

where $z = \lambda h,\ v = qh$.

We can write the system (14) as follows:

$$A_1\underline{S}_i - A_2\underline{S}_{i-m} = A_3\underline{S}_{i-1} + A_4\underline{S}_{i-1-m}, \tag{15}$$

where $\underline{S}_i = (s_{i-3/4}, s_{i-1/2}, s_{i-1/4}, s_i)^T$, $A_1 = (I - zB), A_2 = vB$,

$$B = \begin{bmatrix} \frac{323}{1440} & -\frac{11}{120} & \frac{53}{1440} & -\frac{19}{2880} \\ \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & -\frac{1}{360} \\ \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & -\frac{3}{320} \\ \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 1+\frac{251}{2880}z \\ 0 & 0 & 0 & 1+\frac{29}{360}z \\ 0 & 0 & 0 & 1+\frac{27}{320}z \\ 0 & 0 & 0 & 1+\frac{7}{90}z \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 & 0 & \frac{251}{2880}v \\ 0 & 0 & 0 & \frac{29}{360}v \\ 0 & 0 & 0 & \frac{27}{320}v \\ 0 & 0 & 0 & \frac{7}{90}v \end{bmatrix},$$

and hence we get

$$W(z,v)M_i = G(z,v)M_{i-1}, \tag{16}$$

where

$$\begin{aligned} &M_i = (\underline{S}_i, \underline{S}_{i-m})^T, \quad M_{i-1} = (\underline{S}_{i-1}, \underline{S}_{i-1-m})^T, \\ &W(z,v) = [A_1| - A_2], \quad G(z,v) = [A_3| + A_4]. \end{aligned}$$

Thus by definition, $z = h\lambda, v = hq$ belongs to the region of P-stability (S_P) of our methods. It is clear that $(z,v) \in S_P$ if the eigenvalues $\mu_\ell(z,v), \ell = 1(1)4$ of the generalized eigenvalue problem

$$\mu W(z,v).\underline{x} = G(z,v).\underline{x}, \ \underline{x} \neq 0 \tag{17}$$

lie inside the unit disc, that is, if

$$|\mu_\ell(z,v)| < 1, \ell = 1(1)4. \tag{18}$$

Now, let

$$\begin{aligned} \Pi(\mu,z,v) = det(\mu W(z,v) - G(z,v)) = \frac{\mu^3}{3840}\Big[(\mu-1)\Big(3(z^4+v^4) \\ + (z^2+v^2)(12zv+420) + 18z^2v^2 + 840zv + 3840\Big) \\ - (\mu+1)\Big(50(z^3+v^3) + (z+v)(150zv+1920)\Big)\Big] = 0 \end{aligned} \tag{19}$$

be the characteristic equation of Eq. (17), then we have

$$\begin{aligned}\mu_\ell(z,v) =& 0, \ell = 1(1)3,\\ \mu_4(z,v) =& \Big[3(z^4+v^4)+50(z^3+v^3)+(z^2+v^2)(12zv+420)\\ &+(z+v)(150zv+1920)+18z^2v^2+840zv+3840\Big]\Big/\\ &\Big[3(z^4+v^4)-50(z^3+v^3)+(z^2+v^2)(12zv+420)\\ &-(z+v)(150zv+1920)+18z^2v^2+840zv+3840\Big].\end{aligned} \tag{20}$$

From Eq. (20) we get $|\mu_\ell(z,v)| < 1, \ell = 1(1)4$ for all values of z and v which satisfies $v < -z$, see Figure 1. Since the roots (20) of the characteristic equation (19) are inside the unit circle for $v < -z$, then under the condition $Re(z) < -|v|$ the numerical solution $S_i \longrightarrow 0$ as $i \longrightarrow \infty$. Hence, the methods are P-stable, and a region S_P of P-stability was obtained by determining $\{(z,v) : |v| < -Re(z)\}$. See some region of P-stability in Figure 2.

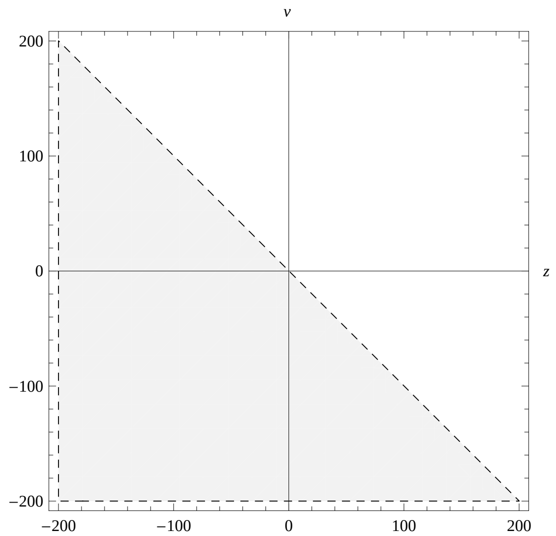

Fig. 1. Some region for values of z, v which satisfies Eq. (18).

5 Numerical examples

To illustrate our discussion, three test examples will be considered. We can compute their actual error and compare the performance of the above mentioned method. The computer application program MATLAB 7.1 was used to execute the algorithms that were used to solve the given examples.

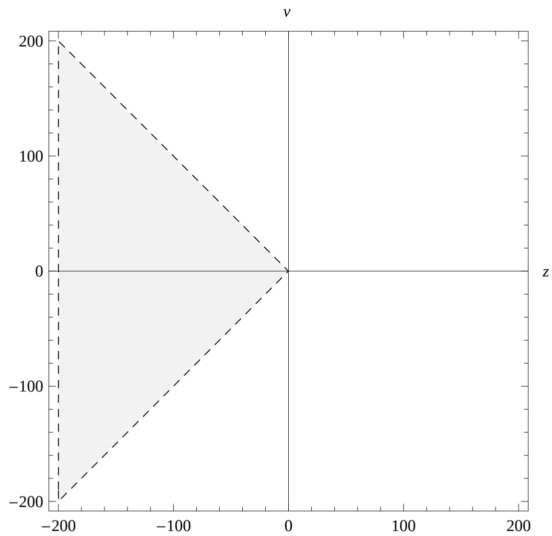

Fig. 2. Some region of P-stability.

Example 5.1. *[7, 12]*

$$\begin{aligned}&\varepsilon y'(x) = -5y(x) + 4y(x-1), \quad x \geq 0.\\&y(x) = e^{-x}, \quad -1 \leq x \leq 0.\end{aligned}$$

The reference solution is: $y(10) = 0.227442715416$ for $\varepsilon = 1$, $y(10) = 0.131413236978$ for $\varepsilon = 0.1$ and $y(10) = 0.10954547852$ for $\varepsilon = 0.01$. In Table 1, we give the errors between the reference solution at $x = 10$ and the computed solution obtained by various methods for Example 5.1. In Figure 3, we plot the numerical solutions obtained with our present method for Example 5.1 with $h = 0.1$ and different values of the parameter ε.

Example 5.2. *[7]*

$$\begin{aligned}&\varepsilon y'(x) + y(x) + \frac{1}{5}y(\frac{x}{2}) = \frac{1}{5}e^{-x/2\varepsilon}, \quad 0 \leq x \leq 1.\\&y(0) = 1, \quad -1 \leq x \leq 0.\end{aligned}$$

Which has the exact solution $y(x) = e^{-x/\varepsilon}$. In Table 2 we give the absolute error between the exact solution and the numerical results by the present method at the end point $x = 1$.

Example 5.3. *[8, 11]*

$$\begin{aligned}&\varepsilon y_1'(x) = -\frac{1}{2}y_1(x) - \frac{1}{2}y_2(x-1) + f_1(x),\\&\varepsilon y_2'(x) = -y_2(x) - \frac{1}{2}y_1(x - \frac{1}{2}) + f_2(x), \quad x \in [0, 5],\end{aligned}$$

Table 1: Absolute errors for the solution of Example 5.1

ε	h	[7]	present method
1	1/20	9.37E-05	6.6323E-12
	1/30	4.14E-05	5.6518E-13
	1/40	2.32E-05	8.6070E-14
0.1	1/20	2.51E-04	5.5844E-14
	1/30	1.17E-04	5.8952E-14
	1/40	9.69E-05	4.6518E-14
0.01	1/20	5.12E-05	4.9081E-10
	1/30	3.25E-05	6.1971E-11
	1/40	2.32E-05	6.1962E-11

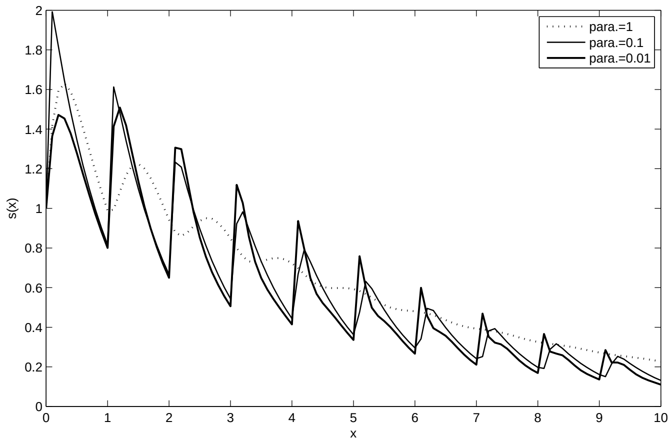

Fig. 3. The numerical solutions of Example 5.1 for $\varepsilon = 1, \varepsilon = 0.1, \varepsilon = 0.01$.

with the initial functions

$$y_1(x) = e^{-x/2\varepsilon}, \quad for \quad -\frac{1}{2} \leq x \leq 0,$$
$$y_2(x) = e^{-x/\varepsilon}, \quad for \quad -1 \leq x \leq 0,$$

and

$$f_1(x) = \frac{1}{2}e^{-(x-1)/\varepsilon}, \quad f_2(x) = \frac{1}{2}e^{-(x-1/2)/2\varepsilon}.$$

The exact solution is given by

$$y_1(x) = e^{-x/2\varepsilon}, \quad y_2(x) = e^{-x/\varepsilon}.$$

Table 2: Test results for Example 5.2

ε	$n = 16$	$n = 32$	$n = 64$
1/16	6.987E-09	1.061E-10	1.648E-12
1/32	1.244E-07	1.460E-09	2.215E-11
1/64	5.592E-07	2.505E-08	2.969E-10

In Table 3, the numbers referred to the maximum absolute error between the exact solution and the approximate solution for different values of x, n, ε.

Table 3: Maximum absolute errors for the solution of Example 5.3

ε	x_i	$n = 16$	$n = 32$	$n = 64$	$n = 128$
1/16	1	1.636E-09	2.580E-11	4.040E-13	6.244E-15
	3	1.241E-08	2.043E-10	3.233E-12	5.063E-14
	5	1.135E-08	1.790E-10	2.803E-12	1.612E-14
1/32	1	6.162E-09	9.518E-11	1.482E-12	2.315E-14
	3	1.994E-07	3.142E-09	4.920E-11	7.692E-13
	5	1.181E-07	1.980E-09	3.147E-11	4.939E-13
1/64	1	3.518E-10	4.989E-12	7.604E-14	1.180E-15
	3	1.481E-08	2.194E-10	3.386E-12	5.274E-14
	5	5.671E-08	8.848E-10	1.383E-11	2.162E-13
1/128	1	2.308E-14	8.806E-17	1.219E-18	1.848E-20
	3	4.755E-13	4.104E-15	5.919E-17	9.084E-19
	5	1.187E-11	7.304E-14	1.027E-15	1.564E-17

6 Conclusion

In this paper, we have developed a collocation method with quintic C^1-splines as basis functions to solve the stiff delay differential equations. Our present methods have convergence of order five. Analysis of stability was also considered. The proposed method is applied to solve two examples of SDDEs with single delay and one example of SDDEs with several delay terms to test the efficiency of the proposed method. Numerical examples have also been used to demonstrate the efficiency and accuracy of the proposed method.

7 Open Problem

We can introduce the analysis of convergence and the stability analysis properties of quintic C^1-spline collocation methods for solving delay partial differential equations.

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