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Classification of uniform polyhedra by their symmetry-type graphs

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Abstract

ABSTRACT: Applying the recently developed theory of flag graphs and k-orbit maps to polyhedra we classify the 75 uniform polyhedra by their symmetry-type graphs. The presented general method and algorithm for finding symmetry-type graphs paves the way for a similar classification of non-uniform polyhedra.

Keywords: *Uniform Polyhedra, Flag Graphs, Symmetry-Type Graphs.*
AMS Mathematics Subject Classifications 2000: 05C78

1 Introduction

The *goal* of this paper is: 1) to present a *classification of the 75 uniform polyhedra based on their symmetry-type graphs* $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of two kinds: the first ones defined by all the isometries of $\mathbb{R}^3$ preserving a given polyhedron $\mathcal{P}$, and the others only by rotations (thus we extend a similar classification of Platonic and Archimedean solids [9]); 2) to present a general *method* and *algorithm* for finding symmetry-type graphs of uniform polyhedra (allowing generalizations to non-uniform polyhedra).

The *main result* (Theorem 1, Section 3) states two basic points:

i) *There are 10 classes of uniform polyhedra* (defined by the 16 different symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of uniform polyhedra);

ii) *The symmetry-type graphs* $T(\mathcal{P})$ *and* $T_R(\mathcal{P})$ *and hence the class* $C(\mathcal{P})$ *of an uniform polyhedron* $\mathcal{P}$ *depend only on its vertex pattern type* (e.g. $(p.q.q)$, $(p.q.p.q)$, $(p.q.r.q.q.q)$ etc.; the notation by vertex pattern is explained in [7]).

The *structure* of the paper follows the usual pattern of research articles (while its *style* of writing exemplifies the creative approach to problems, elaborated in [2]): First we present some basic concepts (Section 2) and formulate the main result (Theorem 1, Section 3); then we discuss the tools, method and algorithm (Section 4), used in the proof of the main result (Section 5), and finally (Section 6) we analyze the solution and list some related open problems.

2 Preliminaries

In this section we define or mention some basic terms, notation and facts regarding polyhedra and their symmetries and explain the concepts of flag graphs and symmetry-type graphs ([14]), on which our classification of polyhedra is built.

POLYHEDRON: During the intense investigation of polyhedra extending over a period of more than 4000 years (e.g. the Moscow Papyrus dated c. 1890 B.C. suggests Egyptians knew how to compute the volume of a truncated square pyramid [19], p.81) accumulated very different, mutually exclusive definitions of the term polyhedron; they were viewed as solids, surfaces or frameworks – depending on the period in which geometers lived and the problems they studied ([4], p.12.). An introduction to polyhedron theory may be found for instance in [19], pp.191-197. For us a *polyhedron* $\mathcal{P}$ will be a solid in Euclidean space $\mathbb{R}^3$ with the given sets of *vertices* $V(\mathcal{P})$, *edges* $E(\mathcal{P})$ and (polygonal or star) *faces* $F(\mathcal{P})$. Two faces $f, g \in F(\mathcal{P})$ are of the same *type*, if they are congruent. The type of a regular polygonal face is denoted simply by the number of its edges $(3, 4, 5, 6, 8, 10, 12$, etc.), the type of a star face is expressed with two numbers $(5/2, 10/3, etc.)$.

UNIFORM POLYHEDRON: A polyhedron $\mathcal{P}$ is called *vertex-transitive* if for any $u, v \in V(\mathcal{P})$ there is a symmetry $h \in I(\mathcal{P})$ such that $h(u) = v$. A polyhedron $\mathcal{P}$ is called *uniform* if it is vertex-transitive and if all its faces are regular polygons or regular stars. Uniform polyhedra can be described by their *vertex pattern* – the cycle of faces around any of their vertices. Some of them, like a snub cube (3.3.3.3.4) have two different forms, being mirror images of each other. A regular-faced polyhedron with only one type of vertex is not necessarily uniform: J.C.P.Miller discovered a non-uniform polyhedron with one vertex type, the same as that of a rhombicuboctahedron (3.4.4.4) ([1], p.137, [7], p.172).

Uniform polyhedra with just one type of face are called *regular polyhedra*. There are five convex regular polyhedra, called the *Platonic solids*, and there are four non-convex regular polyhedra, called the *Kepler-Poinsot polyhedra*.

THE SYMMETRY GROUP $I(\mathcal{P})$: The group of isometries $I(\mathbb{R}^3)$ of Euclidean space $\mathbb{R}^3$ consists of translations, reflections (over a plane, over a point or central reflection, and glide-reflections) and rotations. The *symmetry group* $I(\mathcal{P})$ of polyhedron $\mathcal{P}$, defined as the group of isometries $h \in I(\mathbb{R}^3)$ preserving $\mathcal{P}$, consists of the sets of rotations $Rot(\mathcal{P})$ and reflections (over a plane or a point) $Ref(\mathcal{P})$. The elements of $I(\mathcal{P})$ are called *symmetries* of $\mathcal{P}$.

THE FLAG GRAPH $G_{\mathcal{P}}$ AND THE MONODROMY GROUP $Mon(\mathcal{P})$: If all the faces $f \in \mathcal{P}$ are regular polygons or regular stars, we can make a baricentric subdivision of its faces into triangles, called *flags*. The vertices of any such flag Φ, denoted by Φ_2 (the center of the face f incident with Φ), Φ_1 (the center of the edge e incident with Φ) and Φ_0 (the vertex of the edge e incident with Φ), are called the *face*, the *edge* and the *vertex* of Φ, respectively.

Each flag Φ has three *adjacent flags*, sharing an edge with Φ: the 0-*adjacent flag* Φ^0 lies in the same face f as Φ and along the same edge of f; the 1-*adjacent flag* Φ^1 lies in the same face f as Φ, but not along the same edge of f; the *2-adjacent flag* Φ^2 lies along the same edge of f, but not in the same face as Φ [14], see Figure 1.

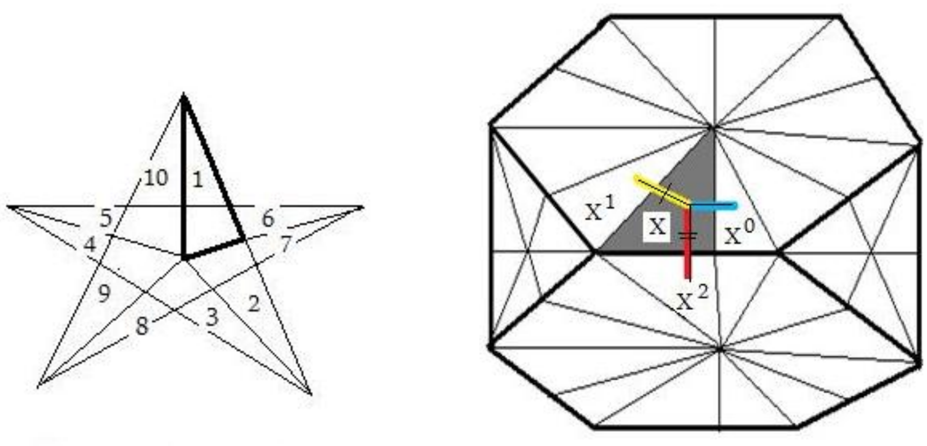

Figure 1: Left – flags in a pentagonal star, right – adjacent flags of X.

The *flag graph* $G_{\mathcal{P}}$ of a polyhedron $\mathcal{P}$ is a graph whose vertex set consists of all the triangles (flags) obtained from the baricentric subdivision of its faces. The edges connecting pairs of adjacent flags (Φ, Φ^0), (Φ, Φ^1), (Φ, Φ^2) are labeled $0, 1$ and 2, respectively. Involutions s_0, s_1 and s_2 of the flag graph, carrying flags $\Phi \in G_{\mathcal{P}}$ into their adjacent flags: $s_0(\Phi) = \Phi^0$, $s_1(\Phi) = \Phi^1$, $s_2(\Phi) = \Phi^2$, satisfy the relations $(s_0s_2)^2 = id = s_0^2 = s_1^2 = s_2^2$. The group $Mon(\mathcal{P})$, generated by s_0, s_1, s_2 is called the *monodromy group* of $\mathcal{P}$.

THE GROUP $Aut(G_{\mathcal{P}})$: Let $Aut(G_{\mathcal{P}})$ denote the group of automorphisms of the flag graph $G_{\mathcal{P}}$, preserving not only adjacency of vertices of $G_{\mathcal{P}}$ but also the labels $0, 1, 2$ of edges. Given any two flags Φ and Ψ of $G_{\mathcal{P}}$, there is at most one automorphism $\widetilde{h} \in Aut(G_{\mathcal{P}})$ carrying a flag Φ into a flag Ψ [14]. Consequently, any symmetry $h \in I(\mathcal{P})$ can be described by an ordered pair $(\Phi, \widetilde{h}(\Phi))$. $I(\mathcal{P})$ is isomorphic to a subgroup $\widetilde{I}(\mathcal{P})$ of $Mon(\mathcal{P})$, since to different isometries $h_1, h_2 \in I(\mathcal{P})$ correspond different automorphisms $\widetilde{h_1}, \widetilde{h_2} \in Aut(G_{\mathcal{P}})$. Thus $\widetilde{h}$ can be denoted by an orderer pair (Φ, Ψ). Clearly $Mon(\mathcal{P})$ is a subgroup of $Aut(G_{\mathcal{P}})$

and $\widetilde{I}(\mathcal{P})$ is a subgroup of $Mon(\mathcal{P})$. Thus $I(\mathcal{P}) \cong \widetilde{I}(\mathcal{P}) \lhd Mon(\mathcal{P}) \lhd Aut(G_{\mathcal{P}})$. Hence any symmetry $h \in I(\mathcal{P})$ can be described by an ordered pair $(\Phi, \widetilde{h}(\Phi))$, where $\widetilde{h} \in Mon(\mathcal{P})$, or better, denoted by $h(\Phi, \widetilde{h}(\Phi))$.

ORBIT OF A FLAG: The *orbit* $T(\Phi)$ of a flag $\Phi \in V(G_{\mathcal{P}})$ is a set of all flags into which Φ is carried by all the isometries $h \in I(\mathcal{P})$ preserving the polyhedron: $T(\Phi) = \{\widetilde{h}(\Phi), h \in I(\mathcal{P})\}$. Each member of the orbit $T(\Phi)$ is called a *representative* of that orbit. Any symmetry of a given polyhedron $\mathcal{P}$ can be simply described by telling which $s \in Mon(\mathcal{P})$ preserves the orbit of a chosen flag Φ of a given face f.

THE SYMMETRY-TYPE GRAPHS $T(\mathcal{P})$ AND $T_R(\mathcal{P})$: The quotient graph of $G_{\mathcal{P}}$ under the action of $\widetilde{I}(\mathcal{P})$ (whose vertices are orbits of flags of $G_{\mathcal{P}}$ and whose edges labeled 0, 1 and 2 correspond to labeled edges of their representatives) is called the *symmetry-type graph* of the polyhedron $\mathcal{P}$ and is denoted by $T(\mathcal{P})$. From this definition immediately follows: $T(s_i(\Phi)) = s_i(T(\Phi))$ for all three involutions s_0, s_1, s_2 of the flag graph. Hence $T(s(\Phi) = s(T(\Phi))$ for any $s \in Mon(\mathcal{P})$ and any $\Phi \in G_{\mathcal{P}}$. For the classification of polyhedra we will use also another type of quotient graph of $G_{\mathcal{P}}$, denoted by $T_R(\mathcal{P})$. Here the orbit $T_R(\Phi)$ of Φ consists only of those flags Ψ, for which there is a rotation $h \in Rot(\mathcal{P})$, carrying Φ into Ψ. If two flags Φ and Ψ lie in the same orbit, we write $\Phi \approx \Psi$ instead of $T(\Phi) = T(\Psi)$ or $\Phi \sim \Psi$ instead of $T_R(\Phi) = T_R(\Psi)$.

PRE-GRAPH, HALF-EDGE: Flag-graphs are 3-regular, while in the symmetry-type graphs there may be loops (edges connecting an orbit with itself. Introducing the concept of *pre-graphs* and using *half-edges* instead of loops [15] we can represent them as 3-regular pregraphs without loops. Half-edges may be also of three types: 0, 1 and 2. When we represent flag graphs and symmetry-type graphs with drawings, we usually label the edges with numbers $0, 1, 2$ or color them blue, yellow and red. We can also mark the 1-edges with the symbol ($|$) and the 2-edges with two parallel lines ($\|$).

T_R-ORBITS AND T-ORBITS: Since we are interested in two different kinds of symmetry-type graphs: $T(\mathcal{P})$ and $T_R(\mathcal{P})$, we will distinguish between the T_R-orbits (defined only by rotations preserving the polyhedron) and the T-orbits (defined also by reflections preserving the polyhedron). We will see that sometimes it is easier to find T_R-orbits and from them deduce what are T-orbits, and sometimes it is just the opposite: we first determine the T-orbits, and from them deduce what must be the T_R-orbits. We may be interested not only in orbits of flags, but also in orbits of vertices, edges and faces, too.

3 Main results

In this section we present a classification of uniform polyhedra by their symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$.

Two polyhedra $\mathcal{P}$ and $\mathcal{Q}$ are in the same equivalence class if they have the same symmetry-type graphs: $T(\mathcal{P}) = T(\mathcal{Q})$ and $T_R(\mathcal{P}) = T_R(\mathcal{Q})$.

We will see that there are 10 equivalence classes of uniform polyhedra and that the equivalence class of any uniform polyhedron $\mathcal{P}$ depends only on its vertex pattern type (Theorem 1).

Figure 2 shows representations of all the 16 symmetry-type graphs of the 75 uniform polyhedra. Some of these graphs are well known [12].

Our notation for them was chosen so that:

i) it indicates the number of their vertices,

ii) it distinguishes between non-isomorphic graphs with the same number of vertices and different 0-edges,

iii) it emphasizes the fact that the graph 6+6 has two 1-2 cycles;

iv) it tries to capture the similarity between symmetry-type graphs of different classes (compare graphs with letters a), b) or c) following the numbers).

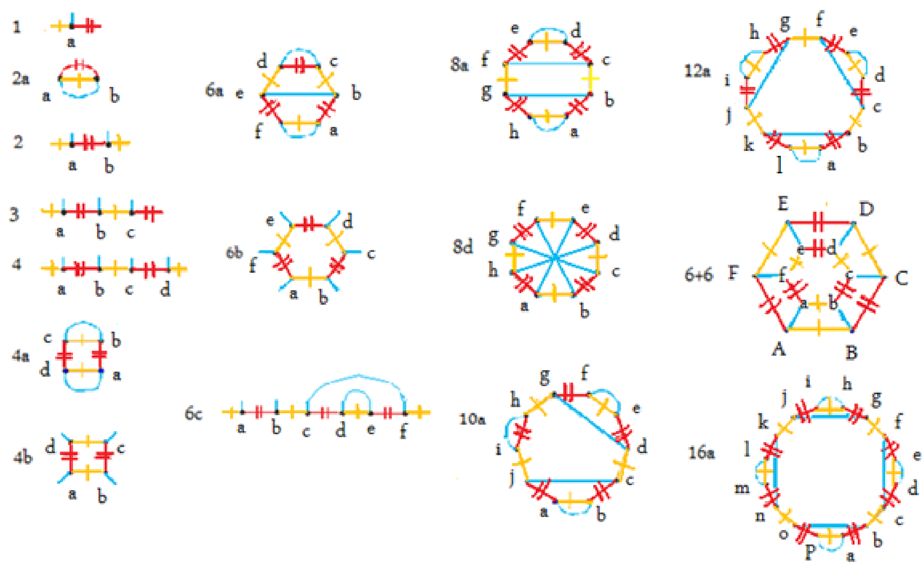

Figure 2: Symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of uniform polyhedra.

The classification of uniform polyhedra by their symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ is presented in the following theorem (Theorem 1), whose proof is given in Section 5.

Theorem 1 *There are 16 different symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of the 75 uniform polyhedra and they have have 1, 2, 3, 4, 6, 8, 10, 12 or 16 vertices (representing orbits of flags in the original flag graph $G_{\mathcal{P}}$).*

These symmetry-type graphs can be described by the permutations of orbits $a, b, c, ...,$ induced by the involutions s_0, s_1, s_2, as follows (see Table 1).

graph	s_0	s_1	s_2
1	id	id	id
2a	(ab)	(ab)	(ab)
2	id	id	(ab)
3	id	(a)(bc)	(ab)(c)
4	id	(a)(bc)(d)	(ab)(cd)
4a	(ad)(bc)	(ad)(bc)	(ab)(cd)
4b	id	(ab)(cd)	(ad)(bc)
6a	(af)(be)(cd)	(af)(bc)(de)	(ab)(cd)(ef)
6b	id	(ab)(cd)(ef)	(af)(bc)(de)
6c	(a)(b)(cf)(de)	(a)(bc)(de)(f)	(ab)(cd)(ef)
8a	(ah)(bg)(cf)(de)	(ab)(cd)(ef)(gh)	(ah)(bc)(de)(fg)
8d	(ae)(bf)(cg)(dh)	(ab)(cd)(ef)(gh)	(ah)(bc)(de)(fg)
10a	(ab)(cj)(dg)(ef)(ih)	(ab)(cd)(ef)(gh)(ij)	(aj)(bc)(de)(fg)(hi)
12a	(al)(bk)(cf)(de)(gj)(hi)	(al)(bc)(de)(fg)(hi)(jk)	(ab)(cd)(ef)(gh)(ij)(kl)
6+6	(aA)(bB)(cC) (dD)(eE)(fF)	(ab)(cd)(ef) (AB)(CD)(EF)	(af)(bc)(de) (AF)(BC)(DE)
16a	(ap)(bo)(cf)(de) (gj)(hi)(kn)(lm)	(ap)(bc)(de)(fg) (hi)(jk)(lm)(no)	(ab)(cd)(ef)(gh) (ij)(kl)(mn)(op)

Table 1: Symmetry-type graphs of uniform solids described by permutations of orbits, induced by the involutions s_0, s_1, s_2.

The symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of uniform polyhedron $\mathcal{P}$ depend only on its vertex pattern type and define the following 10 classes of uniform polyhedra (see Table 2).

class of $\mathcal{P}$ *vertex type of* $\mathcal{P}$	$T(\mathcal{P})$	$T_R(\mathcal{P})$
I.Regular $(p.p.p)$, $(p.p.p.p)$, $(p.p.p.p.p)$ $(p.p.p.p.p)/2$	*1*	2*a*
II.Quasi-regular $(p.q.p.q)$, $(p.q.p^*.q)$, $(p.q.p.q.p.q)$	*2*	4*a*
III. Truncated regular $(p.q.q)$	*3*	*6a*
IV.Versi-quasi regular $(p.q.r.q)$,$(p.q.q.q)$	*4*	*8*
V. Semi-quasi regular $(p.q.p^*.q^*)$	*4b*	*8*
VI.Truncated quasi-regular $(p.q.r)$	*6b*	*6 + 6*
VII.U75 Great Dirhombicosidodecahedron $(p.q.r.q.p^*.q.r^*.q)$	*8a*	*16a*
VIII.Snub quasi-regular with $n = 5$ *faces* $(p.q.q.q.q)$,$(p.q.q.q.q)/2$, $(p.q.r.q.q)$	*10a*	*10a*
IX. Snub quasi-regular with $n = 6$ *faces* *and reflection symmetry* $(p.q.q.q.q.q)$,$(p.q.q.q.q.q)/2$	*6c*	*12a*
X. Snub quasi-regular with $n = 6$ *faces* *without reflection symmetries* $(p.q.r.q.q.q)$, $(p.q.p^*.q.q.q)$	*12a*	*12a*

Table 2: The 10 classes of uniform polyhedra.

Faces denoted by p^*, q^* are of the same type as faces p, q, respectively, but with the opposite orientation. The names of some of these classes are the same as in Johnson's classification of uniform polyhedra by their vertex figure ([8]).

4 Some tools, a method and an algorithm

In this section we present some tools, a method and an algorithm for finding symmetry-type graphs of uniform polyhedra.

4.1 Some tools for finding symmetry-type graphs

CYCLES OF FLAGS OR ORBITS OF FLAGS: Let $G \in \{G_{\mathcal{P}}, T(\mathcal{P}), T_R(\mathcal{P})\}$ be any flag graph or symmetry-type graph of a polyhedron $\mathcal{P}$.

A 1-2 cycle $C_{12}(X_1, X_2 \ldots X_{2m})$ is a cyclical sequence of vertices $X_i \in V(G)$, such that $s_1(X_{2i-1}) = X_{2i}$ and $s_2(X_{2i}) = X_{2i+1(mod\,2m)}$ for any $i \in \{1, 2, \ldots, m\}$.

A 2-0 cycle $C_{12}(X_1, X_2 \ldots X_{2m})$ is a cyclical sequence of vertices $X_i \in V(G)$, such that $s_2(X_{2i-1}) = X_{2i}$ and $s_0(X_{2i}) = X_{2i+1(mod\,2m)}$ for any $i \in \{1, 2, \ldots, m\}$.

A 0-1 cycle $C_{01}(X_1, X_2 \ldots X_{2m})$ is a cyclical sequence of vertices $X_i \in V(G)$, such that $s_0(X_{2i-1}) = X_{2i}$ and $s_1(X_{2i}) = X_{2i+1(mod\,2m)}$ for any $i \in \{1, 2, \ldots, 2m\}$.

Such cycles correspond to vertices, edges and faces of a given polyhedron $\mathcal{P}$. All the 0-2 cycles (corresponding to an edge $e \in E(\mathcal{P})$) have length 4 in the flag graph $G_{\mathcal{P}}$ and length $1, 2$ or 4 in the symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$.

Similar cyclical sequences were used for the classification of homogeneous planar nets [3] and in the study of local flag arrangements of tilings [14].

REFLECTION SYMMETRIES AS TRANSFORMATIONS OF FLAGS: For any flag Φ and for any $i \in \{0, 1\}$ let $S_{\Phi_i} : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection of Euclidean space over a plane orthogonal to the face passing through the common edge of Φ and Φ^i. If S_{Φ_i} preserves $\mathcal{P}$, we can write: $S_{\Phi_i} \in I(\mathcal{P})$.

Proposition 1 *Let $\mathcal{P}$ be a uniform polyhedron with n faces around each vertex. If $S_{\Phi_1} \in I(\mathcal{P})$, then $\mathcal{P}$ remains vertex-transitive even if we forbid reflections, hence $T_R(\mathcal{P})$ has at most $2n$ vertices.*

Proof. Since $\mathcal{P}$ is vertex-transitive there are symmetries $h_1, h_2 \in I(\mathcal{P})$ carrying the flag Φ into a pair of 1-adjacent flags incident with any chosen vertex $v \in V(\mathcal{P})$. One of the symmetries h_1, h_2 is a reflection, the other is a rotation. $\square$

Proposition 2 *Let $\mathcal{P}$ be a polyhedron. If there is a reflection symmetry $S_{\Phi_i} \in I(\mathcal{P})$, it implies $T(\Phi) = T(s_i(\Phi))$. In that case there are half-edges (representing loops) labeled i in the symmetry-type graph $T(G)$.*

Proof. If $S_{\Phi_i}(\mathcal{P}) = \mathcal{P}$ then S_{Φ_i} induces an automorphism of $G_{\mathcal{P}}$, and flags Φ and $s_i(\Phi) = S_i(\Phi)$ belong to the same orbit. $\square$

UNIQUE FACE: A face x is called unique around a given vertex u, if it is the only face of its type incident with u. For example, in a polyhedron (3.4.5.4) the faces 3 and 5 are unique (around each vertex), while the two identical faces 4 are not. Likewise in a polyhedron (10.10.5/2) a pentagram star 5/2 is a unique face, while the two regular 10-gons are not.

Proposition 3 *If a uniform polyhedron $\mathcal{P}$ has a unique face x, then $T(\mathcal{P})$ has either n or $2n$ vertices (representing orbits of flags).*

Proof. Uniform polyhedra are vertex-transitive. If x is a unique face of a uniform polyhedron $\mathcal{P}$ then the pairs of flags incident to any of the vertices of x belong to at most two orbits. Since all the orbits contain the same number of flags, there must be at least n orbits of flags in $T(\mathcal{P})$. And since the number of T-orbits $\#o$ divides $2n$, it is either $\#o = n$ (if there is a reflection symmetry $S_{X_1}(\mathcal{P}) = \mathcal{P}$) or $\#o = 2n$ (if there is no such symmetry). $\square$

ODD FACE: A face $f \in \mathcal{P}$ with an odd number of edges is called an *odd face.*

Proposition 4 *If a uniform polyhedron $\mathcal{P}$ contains an odd unique face x, then along each edge of x there must be flags X and X^1, hence $X^0 = X^1$ and $(T(X))^0 = T(X^1)$. In that case we can find another 0-edge between orbits: $T(X^2)^0 = T(X^0)^2$, since the 0-2 cycles of flags have length 4.*

Proof. An odd unique face contains only two types of flags: X and X^1, which must alternate along the edges. $\square$

POSITION VECTOR: Let $f(X)$ denote the type $3, 4, 5$ or 6 of the face x containing the flag X. The position vector $v(X)$ of a flag X is defined as $v(X) = (f(X), f(s_2(X)))$.

Proposition 5 *If two flags have different position vectors, then they can not lie in the same orbit, and they can not even lie in the 0-adjacent orbits.*

Proof. Any two flags X and Y belonging to the same orbit have the same position vectors: if $T(X) = T(Y)$, then $v(X) = v(Y)$. Any two flags X and Y belonging to a pair of 0-adjacent orbits have the same position vectors: if $T(X) = s_0(T(Y))$, then $v(X) = v(Y)$. $\square$

4.2 A method for finding symmetry-type graphs

Some uniform polyhedra (especially the non-convex ones) are very complex objects and it would be very hard to find their symmetry-type graphs as quotients of flag-graphs, since the flag graphs are almost impossible to draw; so a natural question is: *Can we find symmetry-type graphs of uniform polyhedra by an algebraic method?*

An ideal (algebraical) method would simply *calculate* symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ from some algebraic information uniquely determining $\mathcal{P}$ (for example from its vertex pattern type). Until one finds such a method, we can still get the same results also by more practical (partly empirical) method, combining logical reasoning, observation, combinatorics and looking for additional common characteristics of all members of a given class of polyhedra:

USE LOGICAL REASONING TO AVOID DRAWING LARGE FLAG-GRAPHS: Since every uniform polyhedron $\mathcal{P}$ is vertex-transitive, there is a symmetry $h \in S(\mathcal{P})$ carrying the 1-2 cycle of flags around any $u \in V(\mathcal{P})$ into the 1-2 cycle of flags around any other vertex $v \in V(\mathcal{P})$, hence there are at most $2n$ vertices in $T(\mathcal{P})$ and they form a 1-2 cycle. To find $T(\mathcal{P})$ it is obviously enough to discover which of these vertices lie in the same orbit and how they are connected with 0-edges.

OBSERVE THE SYMMETRIES AND IDENTIFY ORBITS: Looking at polyhedron nets, pictures or 3D-models find for each uniform polyhedron $\mathcal{P}$ separately its rotation and reflection symmetries, especially those with symmetry axes or symmetry planes going through a chosen vertex of $\mathcal{P}$. Thus you can find the exact number of vertices in $T(\mathcal{P})$. You also know whether the part of $T(\mathcal{P})$ without 0-edges is a cycle or a path.

USE COMBINATORIAL TRICKS: To find the 0-edges in $T(\mathcal{P})$ you may use some simple combinatorial tricks: i) draw all possible combinations of 0-edges (there are really only a very few possibilities) and eliminate the impossible cases; ii) use the formula $s_0s_2s_0s_2 = id$ (as explained in Proposition 4); iii) use the fact that some faces are unique (hence they can have only two types of flags), iv) count the flags in different faces (for example flags from a face with 5 edges can lie only in 1, 2 or 10 orbits, flags from a triangular faces can lie only in 1, 2, 3 or 6 orbits).

LOOK FOR ADDITIONAL COMMON CHARACTERISTICS OF ALL MEMBERS OF THE SAME CLASS: Once we forget the »ideal goal« of calculating symmetry-type graphs directly from the vertex pattern type, we can make small but useful discoveries in another direction: for example we may find out that all uniform polyhedra with $n = 6$ faces around each vertex have only odd faces and then we can try to use that information for constructing some detail of their symmetry-type graphs.

4.3 An algorithm for finding symmetry-type graphs

To find symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of an uniform polyhedron $\mathcal{P}$ it suffices to execute the following general procedure:

Algorithm 1 (SYMMETRY-TYPE GRAPHS OF UNIFORM SOLIDS)
(1) Label the 1-2 cycle of flags around a chosen vertex with numbers $1, 2, \ldots, 2n$.
(2) Identify flags of this cycle belonging to the same orbit.
(3) Find the 0-edges between orbits.

This algorithm suggests only *what* to do, not specifying *how* to do it. One possible way is the following:

(1) DRAW A BASIC 1-2 CYCLE: Draw a regular $2n$-gon, label its vertices with numbers $1, 2, \ldots, 2n$ and for each $i \in \{1, \ldots, n\}$ label the edges $(2i-1, 2i)$ with 1 and edges $(2i, 2i+1(mod\, 2n))$ with 2.

(2) IDENTIFY ORBITS: Find out if there are any rotation symmetries with axis going through a vertex or reflection symmetries with a reflection plane going through a vertex.

(3) FIND 0-EDGES: Find out the 0-edges using the following techniques and concepts: an odd unique face, counting the flags in various faces etc.

4.4 Some observations about symmetries of polyhedra

Let us mention here a brief summary some discoveries about symmetries of the 75 uniform polyhedra (obtained by observation of their pictures, 3D-models or polyhedron nets); as shown in Section 5, they allow an easy determination of symmetry-type graphs of whole classes of polyhedra at once.

Many uniform polyhedra have a reflection symmetry S_{P_1}, identifying flags P and P^1 in the face p. This is the case with the classes I, II, III, IV, IX.

If a polyhedron has reflection symmetries S_{X_0} in all its faces x then all the 0-edges in the symmetry-type graph $T(\mathcal{P})$ are half-edges. This is the case with the classes I, II, III, IV, V and VI.

The polyhedra from the classes V and VII have a reflection symmetry identifying pairs of opposite flags i and $i+n$ in the basic 1-2 cycle.

The polyhedra from the classes VIII and X have no reflection symmetries.

5 The proof of Theorem 1

In this section we determine symmetry-type graphs of uniform polyhedra. For each polyhedron we give its uniform notation number Uxy and Wenninger ([21]) notation number Wxy. We will consider the following ten classes of polyhedra:

Class I (REGULAR POLYHEDRA) consists of $N = 9$ polyhedra with vertex pattern $(p.p.p)$, $(p.p.p.p)$, $(p.p.p.p.p)$ and $(p.p.p.p.p)/2$:
the five Platonic solids (subclass I.a)
U01 W01 Tetrahedron (3.3.3)
U05 W02 Octahedron (3.3.3.3)
U06 W06 Cube (4.4.4)
U22 W04 Icosahedron (3.3.3.3.3)
U23 W05 Dodecahedron (5.5.5)
and the four Kepler-Poinsot polyhedra (subclass I.b)
U34 W20 Small Stellated Dodecahedron (5/2.5/2.5/2.5/2.5/2)
U35 W21 Great Dodecahedron (5.5.5.5.5)/2
U52 W22 Great Stellated Dodecahedron (5/2.5/2.5/2)
U53 W41 Great Icosahedron (3.3.3.3.3)/2.

Each of these polyhedra $\mathcal{P}$ has reflection symmetries carrying any chosen flag Φ into its adjacent flags Φ^0, Φ^1, Φ^2, consequently there is only one orbit of flags in $T(\mathcal{P})$. By Proposition 1 $\mathcal{P}$ remains vertex-transitive even if we forbid reflections. Rotations $R_{p,2\pi/p}$ around the center of each face p for the angle $2\pi/p$ ensure that there are only two orbits of flags in $T_R(\mathcal{P})$. All the flags at odd distances from any chosen initial flag Φ (if we define distance between two flags as the smallest number of involutions needed to come from one to the other) lie in the same orbit a and all flags at even distances from Φ lie in the same orbit b of $T_R(\mathcal{P})$. Since the flags at odd distances from Φ are adjacent only to flags at even distances from Φ, the orbits a and b are connected with edges labeled 0, 1 and 2.

Class II (QUASI-REGULAR POLYHEDRA) consists of $N = 16$ polyhedra with vertex pattern $(p.q.p.q)$ $(p.q.p^*.q)$, $(p.q.p.q.p.q)$ or $(p.q.p.q.p.q)/2$. The thirteen solids with $n = 4$ faces around each vertex are (subclass II.a):
U07 W11 Cuboctahedron (3.4.3.4)
U24 W12 Icosidodecahedron (3.5.3.5)
U36 W73 Dodecadodecahedron (5/2.5.5/2.5)
U54 W94 Great Icosidodecahedron (5/2.3.5/2.3)
U03 W68 Octahemioctahedron (3.6.3/2.6)
U04 W67 Tetrahemihexahedron (3.4.3/2.4)
U15 W78 Cubohemioctahedron (4.6.4/3.6)
U49 W89 Small Icosihemidodecahedron (3.10.3/2.10)
U51 W91 Small Dodecahemidodecahedron (5.10.5/4.10)
U62 W100 Small Dodecahemicosahedron (5/2.6.5/3.6)
U65 W102 Great Dodecahemicosahedron (5.6.5/4.6)
U70 W107 Great Dodecahemidodecahedron (5/2.10/3.5/3.10/3)
U71 W106 Great Icosihemidodecahedron (3.10/3.3/2.10/3)

The three solids with $n = 6$ faces around each vertex are (subclass II.b):
U30 W70 Small Ditrigonal Icosidodecahedron (5/2.3.5/2.3.5/2.3)
U41 W80 Ditrigonal Dodecadodecahedron (5/3.5.5/3.5.5/3.5)
U47 W87 Great Ditrigonal Icosidodecahedron (5.3.5.3.5.3)/2.

All these polyhedra have reflection symmetries S_{P_1} and S_{Q_1} and rotation symmetry $R_{v,2\pi/n}$. Hence there are only two orbits of flags, say a (belonging to flags from the faces p) and b (belonging to flags from the faces q). Thus $a^0 = a, b^0 = b, a^1 = a, a^2 = b, b^1 = b$, and $T(\mathcal{P})$ is known.

Now let us find $T_R(\mathcal{P})$! By Proposition 1 all these polyhedra remain vertex-transitive also if we forbid reflections. The rotation symmetry $R_{v,2\pi/n}$ implies the 1-2 part of $T_R(\mathcal{P})$ is a 1-2 cycle consisting of four orbits. Two of them, say a and b, belong to the flags from p and two of them, say c and d, belong to the flags from q, hence: $a^1 = b, b^2 = c, c^1 = d$. Consequently $a^0 = b, c^0 = d$, since there are no half-edges in $T_R(\mathcal{P})$ and since any two 0-adjacent flags lie in the same face.

Class III (TRUNCATED REGULAR POLYHEDRA) consists of $N = 10$ polyhedra with the vertex pattern $(p.q.q)$:
U02 W6 Truncated Tetrahedron (3.6.6)
U08 W7 Truncated Octahedron (4.6.6)
U09 W8 Truncated Cube (3.8.8)
U25 W9 Truncated Icosahedron (5.6.6)
U26 W10 Truncated Dodecahedron (3.10.10)
U19 W92 Stellated Truncated Hexahedron (3.8/3.8/3)
U37 W75 Great Truncated Dodecahedron (5/2.10.10)
U55 W95 Great Truncated Icosahedron (5/2.6.6)
U58 W97 Small Stellated Truncated Dodecahedron (5.10/3.10/3)
U66 W104 Great Stellated Truncated Dodecahedron (3.10/3.10/3).

These polyhedra have reflection symmetries S_{P_1} implying $1^1 \approx 2$, $3^1 \approx 6, 5^1 \approx 6$. Thus there are three orbits of flags, say $a = T(1) = T(2), b = T(3) = T(6), c = T(4) = T(5)$. Obviously $a^0 = a$, hence $b^0 = b$ and $c^0 = c$ and $T(\mathcal{P})$ is known.

Let us now determine $T_R(\mathcal{P})$! By Proposition 1 these polyhedra remain vertex-transitive even if we forbid reflections. Hence there are six orbits, say $a = T_R(1)$, $b = T_R(2)$, $c = T_R(3)$, $d = T_R(4)$, $e = T_R(1)$, $f = T_R(6)$. Consequently $a^0 = b$ and therefore $c^0 = f$, $d^0 = e$, since there are no half-edges in $T_R(\mathcal{P})$ and since any two 0-adjacent flags lie in the same face.

Class IV (VERSI QUASI-REGULAR POLYHEDRA) consists of $N = 14$ polyhedra; twelwe of them (subclass IV.a) have vertex pattern $(p.q.r.q)$:
U13 W69 Small Cubicuboctahedron (3/2.8.4.8)
U14 W77 Great Cubicuboctahedron (3.8/3.4.8/3)

U27 W14 Rhombicosidodecahedron (3.4.5.4)
U31 W71 Small Icosicosidodecahedron (5/2.6.3.6)
U33 W72 Small Dodecicosidodecahedron (5.10.3/2.10)
U38 W76 Rhombidodecadodecahedron (5/2.4.5.4)
U42 W81 Great Ditrigonal Dodecicosidodecahedron (5.10/3.3.10/3)
U43 W82 Small Ditrigonal Dodecicosidodecahedron (5/3.10.3.10)
U44 W83 Icosidodecadodecahedron (5.6.5/3.6)
U48 W88 Great Icosicosidodecahedron (5.6.3/2.6)
U61 W99 Great Dodecicosidodecahedron (3.10/3.5/2.10/3)
U67 W105 Great Rhombicosidodecahedron (3.4.5/3.4)

and two of them (subclass IV.b) have vertex pattern $(p.q.q.q)$:

U10 W13 Rhombicuboctahedron (3.4.4.4)
U17 W85 Great Rhombicuboctahedron (3/2.4.4.4).

Let us first find $T_R(\mathcal{P})$ of these polyhedra. They all have reflection symmetry S_{P_1}. By Proposition 1 they remain vertex-transitive even if we forbid reflections. So the 1-2 part $T_R(\mathcal{P})$ is a 1-2 cycle with eight vertices $1, 2, 3, 4, 5, 6, 7, 8$. All these 14 polyhedra have an odd unique face p, hence $1^0 \approx 2$ and consequently $3^0 = 3^{202} \approx 8$. All the 12 polyhedra from the subclass IV.a have also another unique (not necessarily odd) face r, consequently its flags 5 and 6 are 0-adjacent: $5^0 \approx 6$, hence $4^0 = 4^{202} \approx 7$. The two polyhedra (3.4.4.4) and (3/2.4.4.4) from the subclass IV.b have the same $T_R(\mathcal{P})$, since the other two remaining cases $5^0 \approx 7, 4^0 \approx 8$ and $5^0 \approx 4, 7^0 \approx 8$ lead into contradiction, since they would imply a 0-1 cycle of period 6 in each of the squares having only 8 flags.

From $T_R(\mathcal{P})$ we can find $T(\mathcal{P})$, since the reflection symmetry S_{P_1} implies there are four orbits of flags: $a = T(1) = T(2)$, $b = T(3) = T(8)$, $c = T(4) = T(7)$, $d = T(5) = T(6)$ and all the 0-edges are half edges.

Class V (SEMI QUASI-REGULAR POLYHEDRA)

consists of $N = 7$ polyhedra with vertex pattern $(p.q.p^*.q^*)$:

U18 W86 Small Rhombihexahedron (4.8.4/3.8/7)
U21 W103 Great Rhombihexahedron (4.8/3.4/3.8/5)
U39 W74 Small Rhombidodecahedron (10.4.10/9.4/3)
U50 W90 Small Dodecicosahedron (10.6.10/9.6/5)
U56 W96 Rhombicosahedron (6.4.6/5.4/3)
U63 W101 Great Dodecicosahedron (10/3.6.10/7.6/5)
U73 W109 Great Rhombidodecahedron (10/3.4.10/7.4/3)

Since there are four faces, we have a basic 1-2 cycle of flags $1, 2, 3, 4, 5, 6, 7, 8$ around any chosen vertex u. All these polyhedra have a reflection symmetry C over a plane going through the vertex u and the center of polyhedron carrying p into p^* and q into q^*. Therefore the pair of flags $\{1, 2\}$ from p is carried into the pair of flags $\{5, 6\}$ from p^*, and the pair of flags $\{3, 4\}$ from q is carried into pair of flags $\{7,8\}$ from q^*. Thus $T(\mathcal{P})$ has four orbits, say $a = T(1)$, $= T(2)$, $c = T(3)$, $d = T(4)$, and $a^1 = b$, $b^2 = c$, $c^1 = d$. They all have also reflection symmetries S_{X_0} implying $T(X) = T(X^0)$ for any flag X. Thus the central symmetry C must identify exactly the opposite flags i and $i+4\,(mod\,8)$. Therefore $1 \approx 5$, $2 \approx 6$, $3 \approx 7$, $4 \approx 8$. Hence $1^2 = 8 \approx 4$ and the $T(\mathcal{P})$ is known.

The reflection symmetry C implies that $\mathcal{P}$ remains vertex-transitive even if we forbid reflections. For since all these polyhedra are uniform, there is a symmetry $h \in I(\mathcal{P})$ carrying vertex u into any other vertex v of $\mathcal{P}$, and h is either a reflection or rotation of Euclidean space $\mathbb{R}^3$. If h is a reflection symmetry, then the composition of h and C is a rotation of $\mathbb{R}^3$ preserving $\mathcal{P}$ and carrying u into v.

The 1-2 part of $T_R(\mathcal{P})$ is a 1-2 cycle: $(2i-1)^1 \approx 2i$, $(2i)^2 \approx (2i+1)(mod\,8)$ for all $i \in \{1,2,3,4\}$. The 0-edges in $T_R(\mathcal{P})$ connect exactly the identified pairs of orbits: $(T_R(1))^0 = T_R(5)$, $(T_R(2))^0 = T_R(6)$,$(T_R(3))^0 = T_R(7)$,$(T_R(4))^0 = T_R(8)$.

Class VI (TRUNCATED QUASI-REGULAR POLYHEDRA) consists of $N = 7$ polyhedra with the vertex pattern $(p.q.r)$:

U11 W15 Truncated Cuboctahedron (4.6.8)
U16 W79 Cubitruncated Cuboctahedron (8/3.6.8)
U20 W93 Great Truncated Cuboctahedron (8/3.6.4)
U28 W16 Truncated Icosidodecahedron (4.6.10)
U45 W84 Icositruncated Dodecadodecahedron (10/3.6.10)
U59 W98 Truncated Dodecadodecahedron (10/3.4.10)
U68 W108 Great Truncated Icosidodecahedron (10/3.4.6)

Choose any vertex of the polyhedron. Since there are $n = 3$ faces around each vertex, there is a basic 1-2 cycle of flags $1,2,3,4,5,6$ around it.

Since there are three unique faces around each vertex, there are six different position vectors: $v(1) = (p,r)$, $v(2) = (p,q)$, $v(3) = (q,p)$, $v(4) = (q,r)$, $v(5) = (r,q)$, $v(6) = (r,p)$, thus there are 6 orbits of flags, so there are 6 vertices in $T(\mathcal{P})$. The 1-2 part of $T(\mathcal{P})$ is a 1-2 cycle, since $1^1 \approx 2, 2^2 \approx 3, 3^1 \approx 4, 4^2 \approx 5, 5^1 \approx 6, 6^2 \approx 1$. Since the 0-adjacent flags have the same position vector, and since the flags $1,2,3,4,5,6$ have different position vectors, this implies $X^0 \approx X$ for each flag X and we have half-edges of type 0 in each of the six vertices of $T(\mathcal{P})$.

If we forbid reflections, these polyhedra are no more vertex-transitive? There are two kind of vertices, say black and white, with 1-2 cycles of flags $1,2,3,4,5,6$ and $7,8,9,10,11,12$ around them. The orientations of these cycles in adjacent vertices are opposite. Since 0-adjacent flags must have the same position vectors, we see that $X^0 = X + 6$ $(mod\,12)$.

Class VII (U75) consists of just one uniform polyhedron with the vertex pattern $(p.q.r.q.p^{*}.q.r^{*}.q)$:

U75 W119 Great Dirhombicosidodecahedron (5/2.4.3.4.5/3.4.3/2.4)/2).

It has $n = 8$ faces around each vertex. Hence around each vertex there is a 1-2 cycle consisting of $2n = 16$ flags $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16$.

There is a reflection symmetry of the polyhedron carrying 5/2 into 5/3, identifying flags i and $i + 8(mod\, 16)$. Thus there are 8 orbits of flags.

In all the pentagram stars there are just 2 flags which are different and alternate. The same is true in all triangular faces. Hence flags 1 and 2 alternate in pentagram faces and flags 5 and 6 alternate in the triangle faces. Hence $1^0 \approx 2, 5^0 \approx 6$ and hence $3^0 \approx 8, 4^0 \approx 7$. Thus the symmetry-type graph $T(\mathcal{P})$ is known. The involutions s_0, s_1 and s_2 act on the orbits a, b, c, d, e, f, g, h, i, j like this: $s_0 = (ab)(ch)\ (dg)(ef), s_1 = (ab)(cd)(ef)(gh), s_2 = (ah)(bc)(de)(fg)$. The numbers of orbits of vertices, edges and faces of flags are: $vo = 1, eo = 2, fo = 3$.

Now let us find the symmetry-type graph $T_R(\mathcal{P})$. If we forbid reflections, is the polyhedron still vertex-transitive? Yes, since we have rotation symmetries in each pentagon. Yet now we have 16 orbits of flags, forming a 1-2 cycle. What are the 0-edges? This is easy! $1^0 \approx 2$, hence $3^0 \approx 16$. Likewise: $5^0 \approx 6$ implies $4^0 \approx 7$; $9^0 \approx 10$ implies $8^0 \approx 11$ and $13^0 \approx 14$, hence $12^0 \approx 15$.

Class VIII (Snub quasi-regular polyhedra with $n = 5$ faces) consists of $N = 7$ polyhedra;
five of them (subclass VIII.a) have the vertex pattern $(p.q.q.q.q)$ or $(p.q.q.q.q)/2$:

U12 W17 Snub Cube (4.3.3.3.3)

U29 W18 Snub Dodecahedron (5.3.3.3.3)

U57 W113 Great Snub Icosidodecahedron (5/2.3.3.3.3)

U69 W116 Great Inverted Snub Icosidodecahedron (5/3.3.3.3.3)

U74 W117 Great Inverted Retrosnub Icosidodecahedron (5/2.3.3.3.3)/2

and two of them (subclass IV.b) have the vertex pattern $(p.q.r.q.q)$:

U40 W111 Snub Dodecadodecahedron (5/2.3.5.3.3)

U60 W114 Inverted Snub Dodecadodecahedron (5.3.5/3.3.3)

These polyhedra have no reflection symmetries, therefore for them $T_R(\mathcal{P}) = T(\mathcal{P})$. Since they all have at least one unique face p and rotation symmetry around its center for the angle $2\pi/p$ they must have the same orientation of 1-2 cycles of flags $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ around each vertex, and we can determine 0-edges in their symmetry-type graphs by labeling the 1-2 cycles of ten flags around each vertex in the polyhedron net, starting at the unique face. Each of these two polyhedra has two enantiomorphic forms, which are mirror images of each other, and the same holds for their nets. But their symmetry-type graphs are isomorphic.

We can find $T_R(\mathcal{P}) = T(\mathcal{P})$ also more easily like this: the only two flags 1 and 2 in the unique face p must be 0-adjacent to each other (since $T_R(\mathcal{P})$ has no half-edges): $1^0 \approx 2$, hence $3^0 = 3^{202} \approx 10$. Since all non-unique faces of these 7 polyhedra have 3 or 5 vertices, the 0-1 cycle passing through vertices representing orbits of flags 4, 3, 10 and 9 contains exactly 6 vertices. Since all the 0-2 cycles must have length dividing 4, we can easily see (by eliminating a few impossible cases) that this can happen only in two cases, producing isomorphic graphs: either $9^0 \approx 8$, $4^0 \approx 7$ and $5^0 \approx 6$, or $9^0 \approx 6$, $7^0 \approx 8$ and $5^0 \approx 4$.

Class IX (Snub quasi-regular polyhedra with $n = 6$ faces and reflection symmetries) consists of $N = 2$ polyhedra with vertex pattern $(p.q.q.q.q.q)$ or $(p.q.q.q.q.q)/2$:

U32 W110 Small Snub Icosicosidodecahedron (5/2.3.3.3.3.3)

U72 W118 Small Inverted Retrosnub Icosicosidodecahedron (5/2.3.3.3.3.3)/2.

They both have reflection symmetry S_{P_1}, hence they remain vertex-transitive even if we forbid reflections. Let us first find $T_R(\mathcal{P})$. The flags 1 and 2 in the unique face p have to be 0-adjacent to each other: $1^0 \approx 2$, hence $3^0 = 3^{202} \approx 12$.

Since all non-unique faces of these two polyhedra have 3 vertices, the 0-1 cycle passing through vertices representing orbits of flags 4, 3, 12 and 11 contains exactly 6 vertices. Since all the 0-2 cycles have length dividing 4, and all the 0-1 cycles must contain 2 or 6 vertices we can easily see (by eliminating a few impossible cases) that this can happen only in one case: $4^0 \approx 7$, $5^0 \approx 6$, $8^0 \approx 11$,$9^0 \approx 10$. Thus $T_R(\mathcal{P})$ is known. And from $T_R(\mathcal{P})$ we can easily find $T(\mathcal{P})$, since the reflection symmetry S_{P_1} implies: $1^0 \approx 2$, $3^0 \approx 12$, $4^0 \approx 11$, $5^0 \approx 10$, $6^0 \approx 9$, $7^0 \approx 8$.

Class X (Snub quasi-regular polyhedra with $n = 6$ faces) without reflection symmetries consists of $N = 2$ polyhedra with vertex pattern $(p.q.r.q.q.q)$ or $(p.q.p^*.q.q.q)$:

U46 W112 Snub Icosidodecadodecahedron (5.3.5/3.3.3.3)

U64 W Great Snub Dodecicosidodecahedron (5/3.3.5/2.3.3.3)

There are $n = 6$ faces around each vertex. So we have a 1-2 cycle with 12 flags 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. These two polyhedra have no reflection symmetries. Hence $T(\mathcal{P}) = T_R(\mathcal{P})$ and there are exactly 12 orbits of flags.

Each of these two polyhedra has two faces p and r with 5 vertices and four triangular faces q. Since the number of different flags in a face divides the number of flags in that face, there are only two different flags in faces p and r and there can be only two or six different flags in faces q. As a consequence the 0-adjacent pairs $1^0 \approx 2$ in p and $5^0 \approx 6$ in r imply $3^0 \approx 12$ and $4^0 \approx 7$. Since triangular faces can have only flags from either 2 or 6 different orbits, the 1-2 path passing through vertices $11, 12, 3, 4, 7, 8$ must be a part of a 1-2 cycle of length 6, thus: $8^0 \approx 11$ and $9^0 \approx 10$.

This completes the proof of the Theorem 1.□

6 Summary and Open Problems

This paper is a short summary of a long work on calculating the symmetry-type graphs of the 75 uniform polyhedra which finally led to the classification of uniform polyhedra by their symmetry-type graphs. We have found 10 classes of uniform polyhedra (Figure 3).

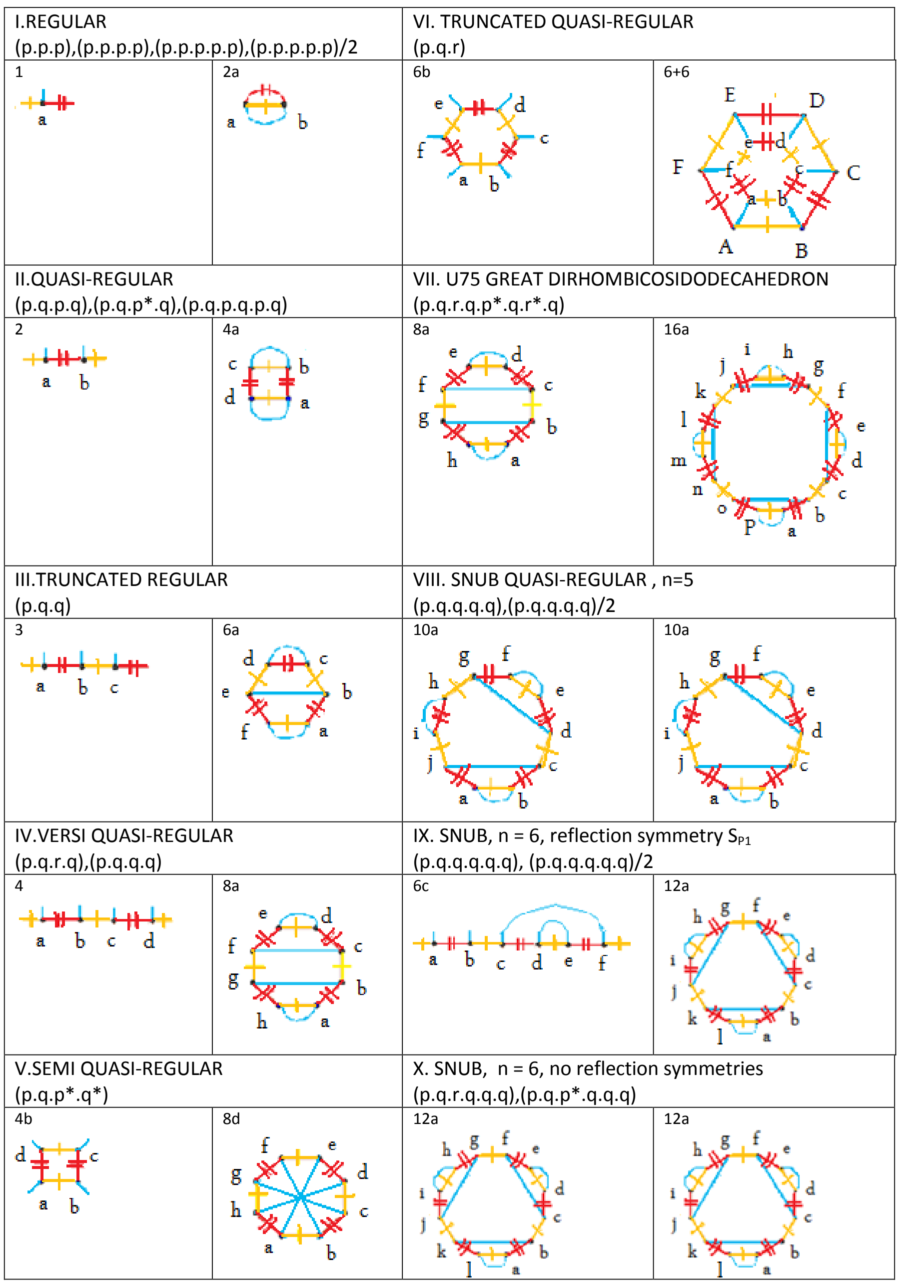

Figure 3: Symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of uniform polyedra.

Our conjecture (»Uniform polyhedra with the same vertex pattern type have the same symmetry type graphs.«), based on the similar classification of Platonic and Archimedean solids [9], finally turned out to be true. We have seen (Theorem 1) that the symmetry-type graphs $T(\mathcal{P})$ and $T_R(\mathcal{P})$ of any uniform polyhedron $\mathcal{P}$ depend only on its vertex type.

Our method and algorithm can be expressed briefly as follows: *Look for rotation and reflection symmetries of any 1-2 cycle of flags, reduce it into a smaller 1-2 cycle or a 1-2 path, and determine the 0-edges.* This method of finding symmetry-type graphs can be adapted to non-uniform polyhedra: *If there are k orbits of vertices, look for rotation and reflection symmetries of 1-2 cycles around representatives of these orbits, reduce these cycles into smaller 1-2 cycles or 1-2 paths, and determine the 0-edges«.*

Some related open problems are: 1) Find the symmetry-type graphs of the 92 Johnson solids; 2) Classify polyhedra with 2 orbits of vertices; 3) Find symmetry-type graphs of the medials $Me(\mathcal{P})$ and truncations $Tr(\mathcal{P})$ of all uniform polyhedra.

This paper builds on the ideas and results from many papers of this area: A classification of edge-transitive maps has been made in [12]. An enumeration of edge-transitive types is given in [5]. The classification of all edge-transitive maps in the torus according to their automorphism group type is given in [20]. Flag graphs first appeared in [10] (there the term »gems«, an acronym for graph-encoded maps, was used). Flag graphs and transformations on maps are discussed in [16]. The question of enumeration of uniform polyhedra is discussed in Problem 26 of [18].

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