Int. J. Open Problems Comput. Math. , Vol. 5, No. 1, March, 2012
ISSN 2074-2827; Copyright ©ICSRS Publication, 2012
www.i-csrs.org

Approximation Of Common Fixed Points Of Generalized Non-Lipschitzian Mappings

R. A. Rashwan

Assiut University, Faculty of Science, Department of Mathematics,Assiut, Egypt
e-mail: rr_ rashwan54@yahoo.com

Abstract

In this paper, we prove strong convergence of a generalized three-step iterative scheme with errors to approximate common fixed points of three generalized Non-Lipschitzian mappings in uniformly convex Banach spaces. Our results generalize and improve several known results.

Keywords: Three-step iterations; Generalized non-Lipschitzian mappings; common fixed points; Uniformly convex;Completely continuous; Demicomapct Mappings

1 Introduction

Throughout this paper, C will denote nonempty convex subset of a real Banach space E and $F(T)$ will denote the set of all fixed points of a mapping T.
A self-mapping $T : C \to C$ is said to be asymptotically nonexpansive [5] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\| T^n x - T^n y \| \leq k_n \| x - y \|,$$

for all $x, y \in C$ and all $n \geq 1$.
The weaker condition (see [9]) requires that

$$\limsup_{n\to\infty} \sup_{y\in C} (\| T^n x - T^n y \| - \| x - y \|) \leq 0,$$

2000 Mathematics Subject Classification: 47H10, 47H09, 46B20.

for each $x \in C$ and that T^N is continuous for some $N \geq 1$.
A definition somewhere between these is follows:
A mapping $T : C \rightarrow C$ is said to be asymptotically nonexpansive in the intermediate sense (see [1]) if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\| T^n x - T^n y \| - \| x - y \|) \quad \leq \quad 0.$$

From the above definitions, it follows that an asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediate sense but the converse is not always true as the following example illustrates:

Example 1.1 *[8] Let $E = R$, $C = [-\frac{1}{\Pi}, \frac{1}{\Pi}]$ and $| \lambda |< 1$. For each $x \in C$, define*

$$T(x) = \begin{cases} \lambda x \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Then T is asymptotically nonexpansive in the intermediate sense but it is not asymptotically nonexpansive.
In 2000, Noor [10] introduced a three-step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [4]used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem,liquid crystal theoery and eigenvalue computation. It has been shown in [4] that the three-step iterative scheme gives better numerical results than the two-step and the one-step approximate iterations. Thus we conclude that the three-step iterative scheme plays an important and significant role in solving various problems arising in pure and applied mathematics.
In 2001, G. E. Kim and T. H. Kim [15] proved the following strong convergence theorem of the Ishikawa iteration process with errors for a non- Lipschitzian self mapping as follows :

Theorem 1.2 *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Suppose $T : C \rightarrow C$ is both completely continuous and asymptotically nonexpansive in the intermediate sense. Put*

$$C_n \quad = \quad \sup_{x,y\in C} (\| T^n x - T^n y \| - \| x - y \|) \vee 0, \quad n \geq 1,$$

so that $\sum_{n=1}^{\infty} C_n < \infty$. For any $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined by the iterative scheme:

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \qquad n \geq 1 \\ y_n &= \acute{\alpha}_n x_n + \acute{\beta}_n T x_n + \acute{\gamma}_n v_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\acute{\alpha}_n\}$, $\{\acute{\beta}_n\}$ and $\{\acute{\gamma}_n\}$ are real sequences in [0,1], $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C such that the following conditions are satisfied :

(i) $\alpha_n + \beta_n + \gamma_n = \acute{\alpha}_n + \acute{\beta}_n + \acute{\gamma}_n = 1,\ n \geq 1.$

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty,\ \sum_{n=1}^{\infty} \acute{\gamma}_n < \infty.$

(iii) $0 < a \leq \alpha_n \leq 1,\ 0 < b_1 \leq \beta_n \leq 1 - b_2 < 1$ *for all* $n \geq 1$ *and some* $a, b_1, b_2 \in (0,1)$.

(iv) $\limsup_{n\to\infty} \acute{\beta}_n \leq b < 1$ *for some* $b \in (0,1)$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a fixed point of* T.

Recently, Xu and Noor [16] introduced and studied a three-step iterative scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach spaces.
In 2004, Cho et al.[2] extended the work of Xu and Noor [16] to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space.
Recently, S. Plubtieng, R. Wangkeeree and R. Punpaeng [12] introduced a three-step iterative scheme with errors for an asymptotically nonexpansive in the intermediate sense mapping and proved the following strong convergence theorem :

Theorem 1.3 *Let* X *be a real uniformly convex Banach space,* C *a nonempty closed convex subset of* X*. Let* T *be a completely continuous asymptotically nonexpansive in the intermediate sense. put*

$$G_n = \sup(\| T^n x - T^n y \| - \| x - y \|) \vee 0, \quad n \geq 1.$$

for all $n \geq 1$*.Let* $x_1 \in C$ *and for each* $n \geq 1$,

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= \alpha_n T y_n + \beta_n x_n + \gamma_n u_n,\ n \geq 1 \\ y_n &= \acute{\alpha}_n T z_n + \acute{\beta}_n x_n + \acute{\gamma}_n v_n, \\ z_n &= \acute{\acute{\alpha}}_n T x_n + \acute{\acute{\beta}}_n x_n + \acute{\acute{\gamma}}_n w_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\acute{\alpha}_n\}$, $\{\acute{\acute{\alpha}}_n\}$, $\{\beta_n\}$, $\{\acute{\beta}_n\}$, $\{\acute{\acute{\beta}}_n\}$, $\{\gamma_n\}$, $\{\acute{\gamma}_n\}$ *and* $\{\acute{\acute{\gamma}}_n\}$ *are appropriate sequences in [0,1] satisfying the following conditions :*

(i) $\alpha_n + \beta_n + \gamma_n = \acute{\alpha}_n + \acute{\beta}_n + \acute{\gamma}_n = \acute{\acute{\alpha}}_n + \acute{\acute{\beta}}_n + \acute{\acute{\gamma}}_n = 1 \quad, n \geq 1.$

(ii) $\{u_n\}$, $\{v_n\}$ *and* $\{w_n\}$ *are bounded sequences in* C.

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty,\ \sum_{n=1}^{\infty} \acute{\gamma}_n < \infty,\ \sum_{n=1}^{\infty} \acute{\acute{\gamma}}_n < \infty.$

(iv) $0 < \alpha \leq \alpha_n, \acute{\alpha}_n, \acute{\acute{\alpha}}_n \leq \beta < 1$ *for some* $\alpha, \beta \in (0,1)$ *and for all* $n \geq n_0$ *for some positive integer* n_0.

Then $\{x_n\}$ converges strongly to a fixed point of T.

Motivated and inspired by the previous results, we introduce and study a new three-step iterative scheme for the approximation of common fixed points of asymptotically nonexpansive in the intermediate sense mappings and establish a strong convergence theorem in uniformly convex Banach spaces. Our results generalize and improve the corresponding results in [2], [3], [6], [7], [8], [11], [12], [13], [16].
The following lemmas and definitions are useful in the sequel :

Definition 1.4 *A mapping $T : C \to C$ is said to be semicompact if, for any bounded sequence $\{x_n\}_{n=1}^{\infty}$ in C such that $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$, there exists a subsequence $\{x_{n_k}\}$, say, of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some x^* in C.*

Lemma 1.5 *[14] Let a_n, b_n, and δ_n be sequences of nonnegative real numbers satisfying the inequality*

$$a_n \leq (1 + \delta_n)a_n + b_n,\ n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 1.6 *[13] Let X be a uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$, and $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\limsup_{n\to\infty} \| x_n \| \leq l$, $\limsup_{n\to\infty} \| y_n \| \leq l$ and $\lim_{n\to\infty} \| t_n x_n + (1 - t_n)y_n \| = l$ for some $l \geq 0$. Then $\lim_{n\to\infty} \| x_n - y_n \| = 0$.*

Let S, T and $R : C \to C$ be three mappings. We will consider the following three-step iterative scheme with errors as follows :

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= \alpha_n S^n y_n + \beta_n x_n + \gamma_n u_n,\ n \geq 1, \\ y_n &= \acute{\alpha}_n T^n z_n + \acute{\beta}_n x_n + \acute{\gamma}_n v_n, \\ z_n &= \acute{\acute{\alpha}}_n R^n x_n + \acute{\acute{\beta}}_n x_n + \acute{\acute{\gamma}}_n w_n, \end{aligned} \tag{1}$$

where $\{\alpha_n\}$, $\{\acute{\alpha}_n\}$, $\{\acute{\acute{\alpha}}_n\}$, $\{\beta_n\}$, $\{\acute{\beta}_n\}$, $\{\acute{\acute{\beta}}_n\}$, $\{\gamma_n\}$, $\{\acute{\gamma}_n\}$ and $\{\acute{\acute{\gamma}}_n\}$ are appropriate sequences in [0,1] satisfying the following conditions :

(i) $\alpha_n + \beta_n + \gamma_n = \acute{\alpha}_n + \acute{\beta}_n + \acute{\gamma}_n = \acute{\acute{\alpha}}_n + \acute{\acute{\beta}}_n + \acute{\acute{\gamma}}_n = 1 \quad, n \geq 1.$

(ii) $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C.

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \acute{\gamma}_n < \infty$, $\sum_{n=1}^{\infty} \acute{\acute{\gamma}}_n < \infty$.

2 Main Results

In this section, we prove strong convergence theorems for the three-step iteration with errors given in (1.1) in real uniformly convex Banach spaces. In order to prove our main results, we need the following lemmas :

Lemma 2.1 *Let C be a nonempty convex subset of a real normed space E. Let S, T and $R : C \to C$ be three asymptotically nonexpansive in the intermediate sense mappings. Put*

$$\begin{aligned}
G_n^{(1)} &= \sup_{x,y\in C}(\| S^n x - S^n y \| - \| x-y \|) \vee 0, \quad n \geq 1,\\
G_n^{(2)} &= \sup_{x,y\in C}(\| T^n x - T^n y \| - \| x-y \|) \vee 0, \quad n \geq 1,\\
G_n^{(3)} &= \sup_{x,y\in C}(\| R^n x - R^n y \| - \| x-y \|) \vee 0, \quad n \geq 1,
\end{aligned}$$

so that $\sum_{n=1}^{\infty} G_n^{(1)} < \infty$, $\sum_{n=1}^{\infty} G_n^{(2)} < \infty$, $\sum_{n=1}^{\infty} G_n^{(3)} < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined in (1.1). If $F = F(S)\cap F(T)\cap F(R) \neq \emptyset$, then $\lim_{n\to\infty} \| x_n - p \|$ exists for each $p \in F$.

Proof Since ,by condition (ii), $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. Then there exists $M > 0$ such that

$$M = \sup_{n\geq 1} \| u_n - p \| \vee \sup_{n\geq 1} \| v_n - p \| \vee \sup_{n\geq 1} \| w_n - p \| .$$

For any $p \in F$, and for all $n \geq 1$, we have

$$\begin{aligned}
\| x_{n+1} - p \| &= \| \alpha_n S^n y_n + \beta_n x_n + \gamma_n u_n - p \|\\
&\leq \alpha_n \| S^n y_n - p \| + \beta_n \| x_n - p \| + \gamma_n \| u_n - p \|\\
&\leq \alpha_n \| y_n - p \| + G_n^{(1)} + \beta_n \| x_n - p \| + \gamma_n M\\
&= \alpha_n \| \acute{\alpha}_n T^n z_n + \acute{\beta}_n x_n + \acute{\gamma}_n v_n - p \| + G_n^{(1)} + \beta_n \| x_n - p \| + \gamma_n M\\
&\leq \alpha_n \acute{\alpha}_n \| T^n z_n - p \| + (\alpha_n \acute{\beta}_n + \beta_n) \| x_n - p \| + G_n^{(1)} + \alpha_n \acute{\gamma}_n \| v_n - p \| + \gamma_n M\\
&\leq \alpha_n \acute{\alpha}_n \| z_n - p \| + (\alpha_n \acute{\beta}_n + \beta_n) \| x_n - p \| + G_n^{(1)} + G_n^{(2)} + (\alpha_n \acute{\gamma}_n + \gamma_n) M\\
&= \alpha_n \acute{\alpha}_n \| \acute{\acute{\alpha}}_n R^n x_n + \acute{\acute{\beta}}_n x_n + \acute{\acute{\gamma}}_n w_n - p \| + (\alpha_n \acute{\beta}_n + \beta_n) \| x_n - p \| +\\
&\quad G_n^{(1)} + G_n^{(2)} + (\alpha_n \acute{\gamma}_n + \gamma_n) M
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \acute{\alpha}_n \acute{\acute{\alpha}}_n \| R^n x_n - p \| + (\alpha_n \acute{\alpha}_n \acute{\acute{\beta}}_n + \alpha_n \acute{\beta}_n + \beta_n) \| x_n - p \| +\\
&\quad \alpha_n \acute{\alpha}_n \acute{\acute{\gamma}}_n \| w_n - p \| + G_n^{(1)} + G_n^{(2)} + (\alpha_n \acute{\gamma}_n + \gamma_n) M\\
&\leq \alpha_n \acute{\alpha}_n \acute{\acute{\alpha}}_n \| x_n - p \| + (\alpha_n \acute{\alpha}_n \acute{\acute{\beta}}_n + \alpha_n \acute{\beta}_n + \beta_n) \| x_n - p \| +\\
&\quad \alpha_n \acute{\alpha}_n \acute{\acute{\gamma}}_n M + G_n^{(1)} + G_n^{(2)} + G_n^{(3)} + (\alpha_n \acute{\gamma}_n + \gamma_n) M,
\end{aligned}$$

which implies that

$$\begin{aligned} \| x_{n+1} - p \| \;\; &\leq \;\; (\alpha_n \acute{\alpha}_n \acute{\acute{\alpha}}_n + \alpha_n \acute{\alpha}_n \acute{\acute{\beta}}_n + \alpha_n \acute{\beta}_n + \beta_n) \| x_n - p \| + \\ &\quad\;\; G_n^{(1)} + G_n^{(2)} + G_n^{(3)} + (\alpha_n \acute{\alpha}_n \acute{\acute{\gamma}}_n + \alpha_n \acute{\gamma}_n + \gamma_n) M, \end{aligned}$$

i. e,

$$\| x_{n+1} - p \| \;\; \leq \;\; \| x_n - p \| + r_n,$$

where $r_n = G_n^{(1)} + G_n^{(2)} + G_n^{(3)} + [\acute{\acute{\gamma}}_n + \acute{\gamma}_n + \gamma_n]M$, $n \geq 1$. Since $\sum_{n=1}^{\infty} G_n^{(1)} < \infty$, $\sum_{n=1}^{\infty} G_n^{(2)} < \infty$, $\sum_{n=1}^{\infty} G_n^{(3)} < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \acute{\gamma}_n < \infty$ and $\sum_{n=1}^{\infty} \acute{\acute{\gamma}}_n < \infty$, then $\sum_{n=1}^{\infty} r_n < \infty$, and, by Lemma 1.3, $\lim_{n\to\infty} \| x_n - p \|$ exists for each $p \in F$.

Lemma 2.2 *Let C be a nonempty convex subset of a real uniformly convex Banach space E. Let S, T and $R : C \to C$ be three asymptotically nonexpansive in the intermediate sense mappings. Let $G_n^{(1)}$, $G_n^{(2)}$, $G_n^{(3)}$ and $\{x_n\}_{n=1}^{\infty}$ be as in Lemma 2.1 with the restriction that $0 < \alpha \leq \alpha_n, \acute{\alpha}_n, \acute{\acute{\alpha}}_n \leq \beta < 1$ for some $\alpha, \beta \in (0,1)$. If $F = F(S) \cap F(T) \cap F(R) \neq \emptyset$, then*

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = \lim_{n\to\infty} \| x_n - S^n y_n \| = \lim_{n\to\infty} \| x_n - T^n z_n \| = \lim_{n\to\infty} \| x_n - R^n x_n \| = 0.$$

Proof For any $p \in F$, $\lim_{n\to\infty} \| x_n - p \|$ exists, by Lemma 2.1.
Suppose $\lim_{n\to\infty} \| x_n - p \| = c$ for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose that $c > 0$. We have

$$\begin{aligned} \| x_{n+1} - p \| \;\; &= \;\; \| \alpha_n S^n y_n + \beta_n x_n + \gamma_n u_n - p \| \\ &= \;\; \| \alpha_n [S^n y_n - p + \gamma_n (u_n - x_n)] + (1 - \alpha_n)[x_n - p + \gamma_n (u_n - x_n)] \|, \end{aligned}$$

which yields that

$$\lim_{n\to\infty} \| \alpha_n [S^n y_n - p + \gamma_n (u_n - x_n)] + (1 - \alpha_n)[x_n - p + \gamma_n (u_n - x_n)] \| \;\; = \;\; c. \tag{2}$$

Moreover,

$$\begin{aligned} \| y_n - p \| \;\; &= \;\; \| \acute{\alpha}_n T^n z_n + \acute{\beta}_n x_n + \acute{\gamma}_n v_n - p \| \\ &\leq \;\; \| x_n - p \| + G_n^{(2)} + G_n^{(3)} + \acute{\acute{\gamma}}_n M. \end{aligned} \tag{3}$$

Taking lim sup in both sides of the inequality (2.2), we get

$$\limsup_{n\to\infty} \| y_n - p \| \;\; \leq \;\; c. \tag{4}$$

Since $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C then, by the existence of
$\lim_{n\to\infty} \| x_n - p \|$, there exists a positive real number $\acute{M}$ such that

$$\acute{M} = \sup_{n\geq 1} \| u_n - x_n \| \vee \sup_{n\geq 1} \| v_n - x_n \| \vee \sup_{n\geq 1} \| w_n - x_n \| .$$

Now, we have

$$\begin{aligned} \| S^n y_n - p + \gamma_n (u_n - x_n) \| &\leq \| S^n y_n - p \| + \gamma_n \| u_n - x_n \| \\ &\leq \| y_n - p \| + G_n^{(1)} + \gamma_n \acute{M}, \end{aligned}$$

hence,

$$\limsup_{n\to\infty} \| S^n y_n - p + \gamma_n (u_n - x_n) \| \leq c. \tag{5}$$

Also,

$$\| x_n - p + \gamma_n (u_n - x_n) \| \leq \| x_n - p \| + \gamma_n \acute{M},$$

which implies that

$$\limsup_{n\to\infty} \| x_n - p + \gamma_n (u_n - x_n) \| \leq c. \tag{6}$$

Applying Lemma 1.4, in view of (2.1), (2.4) and (2.5), we obtain

$$\lim_{n\to\infty} \| x_n - S^n y_n \| = 0. \tag{7}$$

Since

$$\begin{aligned} \| x_{n+1} - x_n \| &= \| \alpha_n S^n y_n + \beta_n x_n + \gamma_n u_n - x_n \| \\ &\leq \alpha_n \| S^n y_n - x_n \| + \gamma_n \| u_n - x_n \| \\ &\leq \beta \| x_n - S^n y_n \| + \gamma_n M, \end{aligned}$$

then, using (2.6), we get

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{8}$$

Since

$$\begin{aligned} \| x_n - p \| &\leq \| x_n - S^n y_n \| + \| S^n y_n - p \| \\ &\leq \| x_n - S^n y_n \| + \| y_n - p \| + G_n^{(1)}, \end{aligned}$$

then, by (2.6), it follows that

$$c = \liminf_{n\to\infty} \| x_n - p \| \leq \liminf_{n\to\infty} \| y_n - p \| . \tag{9}$$

Therefore (2.3) together with (2.8) imply that

$$\lim_{n\to\infty} \| y_n - p \| = c.$$

Again, we have

$$\begin{aligned}\| y_n - p \| &= \| \acute{\alpha}_n T^n z_n + \acute{\beta}_n x_n + \acute{\gamma}_n v_n - p \| \\ &= \| \acute{\alpha}_n [T^n z_n - p + \acute{\gamma}_n (v_n - x_n)] + (1 - \acute{\alpha}_n)[x_n - p + \acute{\gamma}_n (v_n - x_n)] \|,\end{aligned}$$

But $\lim_{n\to\infty} \| y_n - p \| = c$. Then,

$$\lim_{n\to\infty} \| \acute{\alpha}_n [T^n z_n - p + \acute{\gamma}_n (v_n - x_n)] + (1 - \acute{\alpha}_n)[x_n - p + \acute{\gamma}_n (v_n - x_n)] \| = c. \tag{10}$$

Since

$$\begin{aligned}\| z_n - p \| &= \| \acute{\acute{\alpha}}_n R^n x_n + \acute{\acute{\beta}}_n x_n + \acute{\acute{\gamma}}_n w_n - p \| \\ &\leq (\acute{\acute{\alpha}}_n + \acute{\acute{\beta}}_n) \| x_n - p \| + G_n^{(3)} + \acute{\acute{\gamma}}_n M \\ &\leq \| x_n - p \| + G_n^{(3)} + \acute{\acute{\gamma}}_n M,\end{aligned}$$

then,

$$\limsup_{n\to\infty} \| z_n - p \| \leq c. \tag{11}$$

Furthermore,

$$\begin{aligned}\| T^n z_n - p + \acute{\gamma}_n (v_n - x_n) \| &\leq \| T^n z_n - p \| + \acute{\gamma}_n \| v_n - x_n \| \\ &\leq \| z_n - p \| + G_n^{(2)} + \acute{\gamma}_n \acute{M},\end{aligned}$$

hence, by (2.10), it follows that

$$\limsup_{n\to\infty} \| T^n z_n - p + \acute{\gamma}_n (v_n - x_n) \| \leq c. \tag{12}$$

On the other hand,

$$\| x_n - p + \acute{\gamma}_n (v_n - x_n) \| \leq \| x_n - p \| + \acute{\gamma}_n \acute{M},$$

which, on taking lim sup in both sides, implies that

$$\limsup_{n\to\infty} \| x_n - p + \acute{\gamma}_n (v_n - x_n) \| \leq c. \tag{13}$$

Once more, using Lemma 1.4, in presence of (2.9), (2.11) and (2.12), we obtain that

$$\lim_{n\to\infty} \| x_n - T^n z_n \| = 0. \tag{14}$$

Since

$$\begin{aligned}\| x_n - p \| &\leq \| T^n z_n - x_n \| + \| T^n z_n - p \| \\ &\leq \| T^n z_n - x_n \| + \| z_n - p \| + G_n^{(2)},\end{aligned}$$

Using (2.13), we have

$$c = \liminf_{n\to\infty} \| x_n - p \| \leq \liminf_{n\to\infty} \| z_n - p \| . \tag{15}$$

Therefore (2.10) together with (2.14) imply that

$$\limsup_{n\to\infty} \| z_n - p \| \leq c \leq \liminf_{n\to\infty} \| z_n - p \|,$$

which means that,

$$\lim_{n\to\infty} \| z_n - p \| \;\; = \;\; c,$$

i. e,

$$\lim_{n\to\infty} \| \acute{\alpha}_n [R^n x_n - p + \acute{\gamma}_n (w_n - x_n)] + (1 - \acute{\alpha}_n)[x_n - p + \acute{\gamma}_n (w_n - x_n)] \| \;\; = c. \tag{16}$$

Since we have

$$\| R^n x_n - p + \acute{\gamma}_n (w_n - x_n) \| \;\; \leq \;\; \| x_n - p \| + G_n^{(3)} + \acute{\gamma}_n \acute{M},$$

then,

$$\limsup_{n\to\infty} \| R^n x_n - p + \acute{\gamma}_n (w_n - x_n) \| \;\; \leq \;\; c. \tag{17}$$

In addition,

$$\| x_n - p + \acute{\gamma}_n (w_n - x_n) \| \;\; \leq \;\; \| x_n - p \| + \acute{\gamma}_n \acute{M},$$

then,

$$\limsup_{n\to\infty} \| x_n - p + \acute{\gamma}_n (w_n - x_n) \| \;\; \leq \;\; c. \tag{18}$$

Finally, using (2.15), (2.16) and (2.17) and applying Lemma 1.4, we obtain that

$$\lim_{n\to\infty} \| x_n - R^n x_n \| \;\; = \;\; 0. \tag{19}$$

Assertions (2.6), (2.7), (2.13) and (2.18) complete the proof.

Lemma 2.3 *Let C be a nonempty convex subset of a real uniformly convex Banach space E. Let S, T and $R : C \to C$ be three asymptotically nonexpansive in the intermediate sense mappings. Let $G_n^{(1)}$, $G_n^{(2)}$, $G_n^{(3)}$ and $\{x_n\}_{n=1}^{\infty}$ be as in Lemma 2.2. Suppose that $F = F(S) \cap F(T) \cap F(R) \neq \emptyset$. Then, if $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $q \in C$, q must be a common fixed point of S, T and R.*

Proof Since, we have

$$\begin{aligned} \| y_n - x_n \| &= \| \acute{\alpha}_n T^n z_n + \acute{\beta}_n x_n + \acute{\gamma}_n v_n - x_n \| \\ &\leq \beta \| T^n z_n - x_n \| + \acute{\gamma}_n \acute{M}. \end{aligned}$$

Then, by (2.13), we obtain

$$\lim_{n\to\infty} \| y_n - x_n \| = 0. \tag{20}$$

Now, we have

$$\begin{aligned} \| x_n - S^n x_n \| &\leq \| x_n - S^n y_n \| + \| S^n y_n - S^n x_n \| \\ &\leq \| x_n - S^n y_n \| + \| y_n - x_n \| + G_n^{(1)}, \end{aligned}$$

which, using (2.6), (2.19) and the assumption on $G_n^{(1)}$, implies

$$\lim_{n\to\infty} \| x_n - S^n x_n \| = 0. \tag{21}$$

Again, we have

$$\begin{aligned} \| z_n - x_n \| &= \| \acute{\acute{\alpha}}_n R^n x_n + \acute{\acute{\beta}}_n x_n + \acute{\acute{\gamma}}_n w_n - x_n \| \\ &\leq \beta \| x_n - R^n x_n \| + \acute{\acute{\gamma}}_n \acute{M}. \end{aligned}$$

Then, by (2.18), we get

$$\lim_{n\to\infty} \| z_n - x_n \| = 0. \tag{22}$$

Also,

$$\begin{aligned} \| x_n - T^n x_n \| &\leq \| x_n - T^n z_n \| + \| T^n z_n - T^n x_n \| \\ &\leq \| x_n - T^n z_n \| + \| z_n - x_n \| + G_n^{(2)}. \end{aligned}$$

Thus, using (2.13), (2.21) and the assumption on $G_n^{(2)}$, it follows that

$$\lim_{n\to\infty} \| x_n - T^n x_n \| = 0. \tag{23}$$

Finally, we have

$$\begin{aligned} \| x_n - R x_n \| \leq\ & \| x_{n+1} - x_n \| + \| x_{n+1} - R^{n+1} x_{n+1} \| + \| R^{n+1} x_{n+1} - R^{n+1} x_n \| \\ & + \| R^{n+1} x_n - R x_n \| \end{aligned}$$

It follows, from (2.7), (2.18) and uniform continuity of R, that

$$\lim_{n\to\infty} \| x_n - R x_n \| = 0. \tag{24}$$

Similarly, using uniform continuity of S and T in view of (2.20) and (2.22), respectively, we obtain that

$$\lim_{n\to\infty} \| x_n - Sx_n \| = \lim_{n\to\infty} \| x_n - Tx_n \| = 0. \tag{25}$$

Assume that $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $q \in C$, then (2.23) and continuity of R yield that

$$\| q - Rq \| \;=\; 0, \tag{26}$$

i. e, $q = Rq$.
Similarly (2.24) with continuity of S and T imply

$$\| q - Sq \| = \| q - Tq \| = 0, \tag{27}$$

which means that $q = Sq = Tq = Rq$. Hence the proof of the lemma.

Now, we are in a position to state and prove the main theorems of this paper.

Theorem 2.4 *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E. Let S, T and $R : C \to C$ be three asymptotically nonexpansive in the intermediate sense self-mappings. Let $G_n^{(1)}$, $G_n^{(2)}$, $G_n^{(3)}$ and $\{x_n\}_{n=1}^{\infty}$ be as in Lemma 2.2. Let $F = F(S) \cap F(T) \cap F(R) \neq \emptyset$. If at least one of the three mappings R, S and T is completely continuous then $\{x_n\}_{n=1}^{\infty}$ converges strongly a to common fixed point of S, T and R.*

Proof Lemma 2.3 guarantees that if $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $q \in C$, then $q \in F$. Thus, we only need to prove strong convergence of $\{x_n\}_{n=1}^{\infty}$. Since , by Lemma 2.1, $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in C. Let R be completely continuous, then $\{Rx_n\}$ has a subsequence $\{Rx_{n_k}\}$ converging strongly to some $q \in C$. But, from (2.23), it follows that

$$\lim_{k\to\infty} \| x_{n_k} - Rx_{n_k} \| \;=\; 0,$$

therefore, by continuity of R, we have

$$\lim_{k\to\infty} \| x_{n_k} - q \| \;=\; 0,$$

since $q \in F$, then by Lemma 2.1, $\lim_{n\to\infty} \| x_n - q \|$ exists and hence $x_n \to q$ as $n \to \infty$.
Similarly, if S is completely continuous then , by boundedness of $\{x_n\}_{n=1}^{\infty}$, there exists a subsequence $\{Sx_{n_k}\}$ of $\{Sx_n\}$ such that $Sx_{n_k} \to q$, $q \in C$, as $k \to \infty$ and using continuity of S, we get , from (2.24), that $x_n \to q$ as $n \to \infty$.
Finally if T is completely continuous, then the same result is obtained in just a similar way.

Theorem 2.5 *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E. Let S, T and $R : C \to C$ be three asymptotically nonexpansive in the intermediate sense self-mappings. Let $G_n^{(1)}$, $G_n^{(2)}$, $G_n^{(3)}$ and $\{x_n\}_{n=1}^{\infty}$ be as in Lemma 2.2. Let $F = F(S) \cap F(T) \cap F(R) \neq \emptyset$. If at least one of the three mappings R, S and T is semicompact then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of S, T and R.*

Proof Lemma 2.1 yields that $\{x_n\}$ is a bounded sequence. Without loss of generality, suppose that R is semicompact. Since, by (2.23), we have $\lim_{n\to\infty} \| x_n - Rx_n \|= 0$, then there exists a subsequence $\{x_{n_k}\}$, say, of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some x^* in C. We have, from (2.23), that, $\lim_{n\to\infty} \| x_{n_k} - Rx_{n_k} \|= 0$. Since R is continuous, then, $\lim_{n\to\infty} \| x^* - Rx^* \|= 0$, which indicates that $x^* = Rx^*$. Using continuity of S and T, in view of (2.24), we obtain that $x^* = Sx^* = Tx^*$, hence $x^* \in F$. Therefore, by Lemma 2.1, we have $\lim_{n\to\infty} \| x_n - x^* \|$ exists, but $\lim_{n\to\infty} \| x_{n_k} - x^* \|= 0$, thus $x_n \to x^*$ as $n \to \infty$.

Theorem 2.6 *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E. Let S, T and $R : C \to C$ be three asymptotically nonexpansive self-mappings with sequences $\{k_n^{(1)}\}_{n=1}^{\infty}$, $\{k_n^{(2)}\}_{n=1}^{\infty}$ and $\{k_n^{(3)}\}_{n=1}^{\infty}$, respectively, such that $k_n^{(1)}, k_n^{(2)}, k_n^{(3)} \geq 1$, $n \geq 1$ and $\sum_{n=1}^{\infty} k_n^{(1)} < \infty$, $\sum_{n=1}^{\infty} k_n^{(2)} < \infty$, $\sum_{n=1}^{\infty} k_n^{(3)} < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be as in Lemma 2.2. If $F = F(S) \cap F(T) \cap F(R) \neq \emptyset$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of S, T and R if any of the following two conditions is satisfies :*

(i) *If at least one of the three mappings S, T and R is completely continuous.*

(ii) *If at least one of the three mappings S, T and R is semicompact.*

Proof Since every asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediate sense then the proof of Theorem 2.6 is obtained in the proof of Theorems 2.4 and 2.5.

In Theorems 2.4 and 2.5, if S=T=R, then we obtain the following corollary ;

Corollary 2.7 *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E. Let T: $C \to C$ be a asymptotically nonexpansive in the intermediate sense mapping. Put*

$$G_n = \sup_{x,y \in C} (\| T^n x - T^n y \| - \| x - y \|) \vee 0, \quad n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be the three-step iterative sequence with errors defined as follows:

$$x_1 \in C,$$

$$\begin{aligned} x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n u_n, \ n \geq 1, \\ y_n &= \acute{\alpha}_n T^n z_n + \acute{\beta}_n x_n + \acute{\gamma}_n v_n, \\ z_n &= \acute{\acute{\alpha}}_n T^n x_n + \acute{\acute{\beta}}_n x_n + \acute{\acute{\gamma}}_n w_n, \end{aligned} \tag{28}$$

where $\{\alpha_n\}$, $\{\acute{\alpha}_n\}$, $\{\acute{\acute{\alpha}}_n\}$, $\{\beta_n\}$, $\{\acute{\beta}_n\}$, $\{\acute{\acute{\beta}}_n\}$, $\{\gamma_n\}$, $\{\acute{\gamma}_n\}$ *and* $\{\acute{\acute{\gamma}}_n\}$ *are appropriate sequences in [0,1] satisfying the following conditions :*

(i) $\alpha_n + \beta_n + \gamma_n = \acute{\alpha}_n + \acute{\beta}_n + \acute{\gamma}_n = \acute{\acute{\alpha}}_n + \acute{\acute{\beta}}_n + \acute{\acute{\gamma}}_n = 1 \quad , n \geq 1.$

(ii) $\{u_n\}$, $\{v_n\}$ *and* $\{w_n\}$ *are bounded sequences in* C.

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \acute{\gamma}_n < \infty$, $\sum_{n=1}^{\infty} \acute{\acute{\gamma}}_n < \infty$.

(iv) $0 < \alpha \leq \alpha_n, \acute{\alpha}_n, \acute{\acute{\alpha}}_n \leq \beta < 1$

If T *is completely continuous or semicompact, then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a fixed point of* T.

Remark.

(1) Corollary 2.7 contains Theorem 3.3 in [12].

(2) In Corollary 2.7, if T is asymptotically nonexpansive with a sequence $\{k_n\}_{n=1}^{\infty}$ in $[0,1]$ such that $\sum_{n=1}^{\infty} k_n < \infty$, Theorem 2.4 and Corollary 2.5 in [2] are obtained and if , in addition, $\gamma_n = \acute{\acute{\gamma}}_n = \acute{\acute{\gamma}}_n = 0$, $n \geq 1$, we obtain Theorem 2.1 in [16].

3 Open Problem

In this section we should present an open problem.
Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let $T_1, T_2, ..., T_m$ be a finite family of generalized non-Lipschitzian mappings, the following problem arises:

If we use a finite family of generalized non-Lipschitzian mappings instead a generalized non-Lipschitzian, how to defined the new multistep iteration and under what conditions the new iteration converges so that our results in this paper can be as corollaries of new results.

References

[1] R. E. Bruck, T. Kuczumow and S. Reich, Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property, *Colloq. Math.* **65**, (1993)169-179.

[2] Y.J. Cho, H. Zhou, G. Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, *Comput. Math. Appl.* **47** (2004)707-717.

[3] H. Fukhar-ud-din , S. H. Khan, Convergence of two-step iterative scheme with errors for two asymptotically nonexpansive mappings, *Int, J. Math. Sci.* **37**(2004)1956-1971.

[4] R. Glowinski and P. Tallec, Augmented Lagrangian Operator-Splitting Methods in Non-linear Mechanics, SIAM, Philadelphia, (1989).

[5] K. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* **35**(1972)171-174.

[6] Z. Huang, Mann and Ishikawa iterations with errors for asymptotically nonexpansive mappings, *Comput. Math. Appl.* **37**(1999)1-7.

[7] S. H. Khan and W. Takahashi, Approximating common fixed points of two asymptotically nonexpansive mappings, Sci. Math. Jap.53(2001)143-148.

[8] G. E. Kim and T. H. Kim, Mann and Ishikawa iterationss with errors for non- Lipschitzian mappings in Banach spaces, *Comp. and Math. Appl.*, **42**(2001), 1565-1570.

[9] W. A. Kirk, Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type, *Israel J. Math.* **17**,(1974) 339-346.

[10] M. A. Noor, New approximation Schemes for general variational inequalities, *J. Math. Anal. Appl.***251**(2000)217-229.

[11] S. Plubtieng, R. Wangkeeree, R. Punpaeng, On the convergence of modified Noor iterations with errors for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.***322**(2006)1018-1029.

[12] S. Plubtieng and R. Wangkeeree, Noor Iterations with Error for Non-Lipschitzian Mappings in Banach Spaces, *Kyungpook Math. J.* **46**(2006), 201-209.

[13] J. Schu, Weak and strong convergence of fixed points of asymptotically nonexpansive mappings, *Bull. Austral. Math. Soc.* **43** (1991) 153-159.

[14] K. K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.* **178**(1993)301-308.

[15] Y. Xu, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator eqnations, *J. Math. Anal. Appl.* **224**(1998)91-101.

[16] B. L. Xu, M. A. Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* **267** (2002) 444-453.