Int. J. Open Problems Compt. Math., Vol. 5, No. 1, March 2012
ISSN 1998-6262; Copyright © ICSRS Publication, 2012
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On Optimality of Lacunary Interpolation for Recovery of C^6 Seventh Degree Spline

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Abstract

In this paper, we constructed seventh splines of deficiency seven used it for the solution of initial value problems. The convergence analysis of the given method is investigated. Numerical illustrations are given to show the applicability and efficiency of our construction.

Keywords: *Spline functions, fifth order initial value problem, convergence analysis.*

AMS subject classifications: *65D05, 65D07 and 65D32.*

1 Introduction

We consider the fifth order initial value problems of the form:

$$y^{(5)}(x) = f(x, y(x), y'(x), y''(x), y'''(x), y^{(4)}(x)) \ , \ x_0 \in [0, 1],$$
$$y(x_0) = y_1 \ , y'(x_0) = y'_2, \ y''(x_0) = y''_3, \ y'''(x_0) = y'''_4, \ y^{(4)}(x_0) = y_5^{(4)} \qquad (1)$$

With the help of lacunary spline functions of type (0, 1, 4) see Jwamer [9], by using that $f \in C^{n-1}([0,1] \times R^2)$, $n \geq 2$ and that it satisfies the Lipschitz continuous

$\left| f^{(q)}(x, y_1, y_1', \ldots, y_1^{(m-1)}) - f^{(q)}(x, y_2, y', y_2'', \ldots, y_2^{(m-1)}) \right| \le L \sum_{i=1}^{m-1} \left| y_1^{(i)} - y_2^{(i)} \right|$, q=0, 1,…, n-1. (2)

for all $x \in [0, 1]$ and all real y_1, y_2, y_1', y_2' form Gyorvari [3]. These conditions ensure the existence of unique solution of the problem (1). The analytic solution of (1) for arbitrary choices of f(x) cannot be found in general. We usually resort to some numerical method for obtaining an approximate solution of the problem (1). The standard numerical methods for the numerical treatment of (1) consist of Taylor's method, Euler's method, finite difference methods, collocation methods. A long list of references of all of these methods is given by [1-2]. Since then many papers have appeared dealing with the continuous approximation of y(x) satisfying (1) via cubic, quantic and sixth splines mainly (see [3-5]).

We recall the basics of brief description method in Section 2 as a preliminary. The derivation of the difference schemes spline function has been given in Section 3, and also, we have shown the convergence analysis are studied and then prove that the interpolation problem is constructible with respect theorem1 . We have solved two numerical examples to demonstrate the applicability of the methods with the new algorithm in section 4. In the last section, the discussion on the results is given in Section 5.

2 The spline function $\bar{S}_\Delta(x)$

Using these approximate values $\overline{y}_i^{(q)}$ $(q = 0, 1, 4.,\ i = 0, 1, 2, \ldots, m)$ and $\bar{y}_0'', \bar{y}_0'''$ on the bases of Jwamer [ ], we construct the lacunary spline function $\bar{S}_\Delta(x)$ of the type (0, 1, 4), $(\bar{S}_\Delta(x) = \bar{S}_i(x)$ if $x_i \le x \le x_{i+1})$ and denote by $\bar{S}_{n,7}^{8}$ the class of seven degree splines $\bar{S}_\Delta(x)$ as the following:

$$\bar{S}_\Delta(x) = \begin{cases} \bar{S}_0(x_i) = \bar{y}_i, \text{ when } x \in [x_0, x_1], \\ \bar{S}_i^{(q)}(x_i) = \bar{y}_i^{(q)}, \text{ when } x \in [x_i, x_{i+1}],\ q = 1, 4. \\ \bar{S}_0''(x_0) = y_0'', \bar{S}_0'''(x_0) = y_0''' \text{ and } \bar{S}_{n-1}(x_n), \text{ when } x \in [x_{n-1}, x_n]. \end{cases} \tag{3}$$

Where $q = 0, 1, 4$, are known derivatives and $i = 0, 1, 2, \ldots, m$, the existence and uniqueness of the above spline function have been shown in [7],

$$\bar{S}_0 = \bar{y}_0 + (x - x_0)\bar{y}_0' + \frac{(x - x_0)^2}{2}\bar{y}_0'' + \frac{(x - x_0)^3}{6}\bar{y}_0''' + \frac{(x - x_0)^4}{24}\bar{y}_o^{(4)} + (x - x_0)^5\bar{a}_{0,5} + (x - x_0)^6\bar{a}_{0,6} + (x - x_0)^7\bar{a}_{0,7} \tag{4}$$

Let us examine now intervals $[x_i, x_{i+1}]$, i=1, 2, …, n-2., Defined $\bar{S}_i(x)$ as:

$$\bar{S}_i(x) = \bar{y}_i + (x-x_i)y_i' + (x-x_i)^2\bar{a}_{i,2} + (x-x_i)^3\bar{a}_{i,3} + \frac{(x-x_i)^4}{24}\bar{y}_i^{(4)} + (x-x_i)^5\bar{a}_{i,5} + (x-x_i)^6\bar{a}_{i,6} + (x-x_i)^7\bar{a}_{i,7} \quad (5)$$

Here from equation (4), (5) and the polynomial coefficients in [7], we can find the following coefficients

$$\bar{a}_{0,5} = \frac{21}{2h^5}(\bar{y}_1 - \bar{y}_0) - \frac{1}{2h^4}(4\bar{y}_1' - 17\bar{y}_0') + \frac{1}{240h}(\bar{y}_1^{(4)} - \bar{y}_0^{(4)}) - \frac{13}{4h^3}\bar{y}_0'' - \frac{3}{4h^2}\bar{y}_0^{(3)}$$

;

$$\bar{a}_{0,6} = \frac{-14}{h^6}(\bar{y}_1 - \bar{y}_0) + \frac{1}{h^5}(3\bar{y}_1' + 11\bar{y}_0') - \frac{1}{120h^2}(\bar{y}_1^{(4)} - 11\bar{y}_0^{(4)}) + \frac{4}{h^5}\bar{y}_0'' + \frac{5}{6h^4}\bar{y}_0^{(3)}$$

;

and

$$\bar{a}_{0,7} = \frac{9}{2h^7}(\bar{y}_1 - \bar{y}_0) - \frac{1}{2h^6}(2\bar{y}_1' + 7\bar{y}_0') + \frac{1}{240h^3}(\bar{y}_1^{(4)} - 6\bar{y}_0^{(4)}) - \frac{5}{4h^5}\bar{y}_0'' - \frac{1}{4h^4}\bar{y}_0^{(3)}$$

also

$$\bar{a}_{1,2} = \frac{-21}{2h^2}(\bar{y}_1 - \bar{y}_0) + \frac{1}{2h}(8\bar{y}_1' + 13\bar{y}_0') + \frac{h^2}{240}(\bar{y}_1^{(4)} + 4\bar{y}_0^{(4)}) + \frac{7}{4}\bar{y}_0'' + \frac{h}{4}\bar{y}_0^{(3)}$$

;

$$\bar{a}_{1,3} = \frac{-35}{2h^3}(\bar{y}_1 - \bar{y}_0) + \frac{1}{2h^2}(10\bar{y}_1' + 25\bar{y}_0') + \frac{h}{48}(\bar{y}_1^{(4)} + 2\bar{y}_0^{(4)}) + \frac{15}{4h}\bar{y}_0'' + \frac{7h^2}{12}\bar{y}_0^{(3)}$$

.

We can find the approximate coefficients in intervals $[x_i,\ x_{i+1}]$, i=1, 2, …, n-2., to defined $\bar{S}_i(x)$ as [9].

$$\bar{a}_{i,5} = \frac{21}{2h^5}(\bar{y}_{i+1} - \bar{y}_i) - \frac{1}{2h^4}(4\bar{y}_{i+1}' - 17\bar{y}_i') + \frac{1}{240h}(\bar{y}_{i+1}^{(4)} - \bar{y}_i^{(4)}) - \frac{13}{2h^3}a_{i,2} - \frac{9}{2h^2}a_{i,3}$$

$$\bar{a}_{i,6} = \frac{-14}{h^6}(\bar{y}_{i+1} - \bar{y}_i) + \frac{1}{h^5}(3\bar{y}_{i+1}' + 11\bar{y}_i') - \frac{1}{120h^2}(\bar{y}_{i+1}^{(4)} - 11\bar{y}_i^{(4)}) + \frac{8}{h^4}a_{i,2} + \frac{5}{h^3}a_{i,3}$$

;

and

$$\bar{a}_{i,7} = \frac{9}{2h^7}(\bar{y}_{i+1} - \bar{y}_i) - \frac{1}{2h^6}(2\bar{y}_{i+1}' + 7\bar{y}_i') + \frac{1}{240h^3}(\bar{y}_{i+1}^{(4)} - 6\bar{y}_i^{(4)}) - \frac{5}{2h^5}a_{i,2} - \frac{3}{2h^4}a_{i,3}$$

also

$$2\bar{a}_{i+1,2} - 7\bar{a}_{i,2} - 3h\bar{a}_{i,3} = \frac{-21}{h^2}(\bar{y}_{i+1} - \bar{y}_i) + \frac{1}{h}(8\bar{y}'_{i+1} + 13\bar{y}'_i) + \frac{h^2}{120}(\bar{y}_{i+1}{}^{(4)} + 4\bar{y}_i{}^{(4)})$$

;

$$6\bar{a}_{i+1,3} - 21\bar{a}_{i,3} - \frac{45}{h}\bar{a}_{i,2} = \frac{-105}{h^3}(\bar{y}_{i+1} - \bar{y}_i) + \frac{15}{h^2}(2\bar{y}'_{i+1} + 5\bar{y}'_i) + \frac{h^2}{48}(\bar{y}_{i+1}{}^{(4)} + 4\bar{y}_i{}^{(4)})$$

.

Similarly for the last interval $[x_{n-1}, x_n]$, we can define approximate values of $\bar{S}_n(x)$.

3 Theoretical Scheme

The new approximate spline function $\bar{S}_\Delta(x)$ given in the section before to the exact solution of the fourth order initial value problem (1) and corresponding to the values of $y_i\,(i = 0,1,2,....,m)$ of a problem (1), and prove the following theorem:

Theorem 1: Let $\bar{y}_i^{(q)}$ $(q = 0,1,\ 4;\ i = 0,1,2...,m)$ be the approximate values defined above. Then the following estimates of spline function $\bar{S}_\Delta(x)$ are valid:

(i) $\left|S_i^{(q)}(x) - \bar{S}_i^{(q)}(x)\right| \le \begin{cases} \bar{G}_q\, h^{8-q}\, \omega_7(h); \text{for } q = 0,1,....,8,\ i = 0\ . \\ \bar{T}_q\, h^{8-q}\, \omega_7(h); \text{for } q = 0,1,....,8,\ i = 1,...,m-2\ . \end{cases}$

where $\bar{G}_q$ and $\bar{T}_q$ denote the difference constants dependent of h.

(ii) $\left|y_i^{(q)}(x) - \bar{S}_i^{(q)}(x)\right| \le U_q\, h^{8-q}\, \omega_7(h); \text{for } q = 0,....,8,$ where $y(x)$ is a solution of problem (1) and D_q denote the difference constants dependent of h.

Proof: (i) From theorem 1 of [8] and equation (3), we have

$$S_0(x) - \bar{S}_0(x) = (x - x_0)^5(a_{0,5} - \bar{a}_{0,5}) + (x - x_0)^6(a_{0,6} - \bar{a}_{0,6}) + (x - x_0)^7(a_{0,7} - \bar{a}_{0,7}) \qquad (6)$$

Where

$$a_{0,5} - \bar{a}_{0,5} = \frac{21}{2\,h^5}(y_1 - \bar{y}_1) - \frac{2}{h^4}(y'_1 - \bar{y}'_1) + \frac{1}{240\,h}(y_1{}^{(4)} - \bar{y}_1{}^{(4)});$$

implies that

$$\left|a_{0,5}-\bar{a}_{0,5}\right| \le \frac{1}{240\,h^5}\left(C_1+480C_2+2520C_3\right)w_7(h)$$

$$=\frac{1}{240}H_0\ \omega_7(h)$$

where $H_1 = C_1 + 120C_2 + 2520C_3$ and C_1 , C_2 and C_3 are constants dependent of h.

$$a_{0,6}-\bar{a}_{0,6} = \frac{-14}{h^6}(y_1-\bar{y}_1)+\frac{3}{h^5}(y_1'-\bar{y}_1')-\frac{1}{120h^2}(y_1^{(4)}-\bar{y}_1^{(4)})$$;

implies that

$$\left|a_{0,6}-\bar{a}_{0,6}\right| \le \frac{1}{120h^6}\left(C_4+360C_5+1680C_6\right)w_7(h)$$

$$=\frac{1}{120}H_1\,w_7(h)$$

where $H_2 = C_4 + 360C_5 + 1680C_6$ and C_4 , C_5 and C_6 are constants dependent of h.

$$a_{0,7}-\bar{a}_{0,7} = \frac{9}{2h^7}(y_1-\bar{y}_1)-\frac{1}{h^6}(y_1'-\bar{y}_1')-\frac{1}{240\,h^3}(y_1^{(4)}-\bar{y}_1^{(4)})$$

implies that

$$\left|a_{0,7}-\bar{a}_{0,7}\right| \le \frac{1}{240h^7}\left(C_7+240C_8+1080C_9\right)w_7(h)$$

$$=\frac{1}{240}H_2\,w_7(h)$$

where $H_3 = C_7 + 360C_8 + 1680C_9$ and C_7 , C_8 and C_9 are constants dependent of h.

And hence

$$\left|S_0(x)-\bar{S}_0(x)\right| \le h^5\left|a_{0,5}-\bar{a}_{0,5}\right|+h^6\left|a_{0,6}-\bar{a}_{0,6}\right|+h^7\left|a_{0,7}-\bar{a}_{0,7}\right|$$

$$\le \bar{G}_0\ \omega_6(h)$$

Where $\bar{G}_0 = H_0 + H_1 + H_2$, dependent of h.

By taking the derivatives of equation (4), we obtain the following:

$$\left|S_0'(x)-\bar{S}_0'(x)\right| = y_1'-\bar{y}_1'$$

which clear that from (3) is known,

$$\left|S_0''(x)-\bar{S}_0''(x)\right| \le \frac{21}{h^2}\left|\bar{y}_1-y_1\right|+\frac{8}{h}\left|y_1'-\bar{y}_1'\right|+\frac{h^2}{120}\left|y_1^{(4)}-\bar{y}_1^{(4)}\right|$$

$$\le \frac{1}{120}(\bar{C}_1+2520\bar{C}_2+960\bar{C}_3)\,\omega_7(h)=\frac{1}{120}\bar{G}_1\omega_7(h);$$

$$\left|S_0^{(3)}(x)-\bar{S}_0^{(3)}(x)\right| \le \frac{105}{h^3}\left|\bar{y}_1-y_1\right|+\frac{30}{h^2}\left|y_1'-\bar{y}_1'\right|+\frac{h}{8}\left|y_1^{(4)}-\bar{y}_1^{(4)}\right|$$

$$=\frac{1}{8}(\bar{C}_1+840\bar{C}_2+240\bar{C}_3)\,\omega_7(h)=\frac{1}{8}\bar{G}_2\omega_7(h)$$

and by successive differentiations obtain

$$\left|S_0^{(q)}(x)-\bar{S}_0^{(q)}(x)\right| \le \bar{G}_q\, h^q\, \omega_7(h);\text{for } \; q=3,4,\ldots,8.$$

This proves (i) for $k=0$ and $x\in[x_0,\, x_1]$. Further more in the interval $[x_{i-1}\,,\, x_i]$

$$\bar{S}_i(x)=\bar{y}_i+(x-x_i)y_i'+(x-x_i)^2\bar{a}_{i,2}+(x-x_i)^3\bar{a}_{i,3}+\frac{(x-x_i)^4}{24}\bar{y}_i^{(4)}+$$
$$(x-x_i)^5\bar{a}_{i,5}+(x-x_i)^6\bar{a}_{i,6}+(x-x_i)^6\bar{a}_{i,6}$$

Similarly, it's clear that, to show the following:

$$2(a_{i+1,2}-\bar{a}_{i+1,2})=7(a_{i,2}-\bar{a}_{i,2})+3h(a_{i,3}-\bar{a}_{i,3})-\frac{21}{h^2}(y_{i+1}-\bar{y}_{i+1})$$
$$+\;\frac{8}{h}(y_{i+1}'+\bar{y}_{i+1}')+\frac{h^2}{120}(y_{i+1}^{(4)}-\bar{y}_{i+1}^{(4)})$$

;

implies that

$$\left|a_{i+1,2}-\bar{a}_{i+1,2}\right| \le \frac{1}{240h^2}\left(C_{10}+120C_{11}+360C_{12}+2520C_{13}+480C_{14}\right)w_7(h)$$

$$=\frac{1}{240}\bar{H}_0 w_7(h)$$

Where $\bar{H}_0=C_{10}+120C_{11}+360C_{12}+2520C_{13}+480C_{14}$, and C_{10} , C_{11}, C_{12}, C_{13} and C_{14} are constants dependent of h.

$$6(a_{i+1,3}-\bar{a}_{i+1,3})=21(a_{i,3}-\bar{a}_{i,3})+\frac{45}{h}(a_{i,2}-\bar{a}_{i,2})-\frac{105}{h^3}(y_{i+1}-\bar{y}_{i+1})$$
$$+\ \frac{30}{h^2}(y'_{i+1}-\bar{y}'_{i+1})+\frac{h^3}{120}(y^{(4)}_{i+1}-\bar{y}^{(4)}_{i+1})$$

;

implies that

$$\left|a_{i+1,3}-\bar{a}_{i+1,3}\right|\le\frac{1}{720h^3}\left(C_{15}+2520C_{16}+5400C_{17}+12600C_{18}+3600C_{19}\right)w_7(h)$$
$$=\frac{1}{720}\bar{H}_1\,w_7(h)$$

where $\bar{H}_1=C_{15}+2520C_{16}+5400C_{17}+12600C_{18}+3600C_{19}$

and C_{15} , C_{16}, C_{17}, C_{18} and C_{19} are constants dependent of h.

$$\left|a_{i,5}-\bar{a}_{i,5}\right|\le\frac{1}{240h^5}\left(C_{20}+1560C_{21}+1080C_{22}+2520C_{23}+480C_{24}\right)w_7(h)$$
$$=\frac{1}{240}\bar{H}_2\,w_7(h)$$

where $\bar{H}_2$ and C_{20}, C_{21}, C_{22}, C_{23} and C_{24} be a constants dependent of h.

And also

$\left|a_{i,6}-\bar{a}_{i,6}\right|\le\bar{H}_3\ \omega_7(h)$;

$\left|a_{i,7}-\bar{a}_{i,7}\right|\le\bar{H}_4\ \omega_7(h)$, where $\bar{H}_4, \bar{H}_6, \bar{H}_7$ and $\bar{H}_8$ are dependent of h.

and by taking the successive differentiation, we can find $\bar{T}_q$ where $q=0,1,....,8$; similarly as before

$\left|S_i^{(q)}(x)-\bar{S}_i^{(q)}(x)\right|\le\bar{T}_q\,h^{8-q}\,\omega_7(h)$; for $q=0,1,....,8.$ Which is prove (i) for $i=1,...,m-2.$

We can repeat the same manner in above for $i=m-1$.

Proof of theorem 1 (ii):

$$\left|y^{(q)}(x)-\bar{S}_\Delta^{(q)}(x)\right|\le\left|y^{(q)}(x)-S_\Delta^{(q)}(x)\right|+\left|S_\Delta^{(q)}(x)-\bar{S}_\Delta^{(q)}(x)\right|$$

From theorem 2 of [5], the following estimates are valid

$$\left|y^{(q)}(x)-\bar{S}_\Delta^{(q)}(x)\right|\le C_q h^{8-q}\omega_7(h) \qquad (7)$$

Using equation (7) and estimate in (i), we have

$$\begin{aligned}\left|y^{(q)}(x)-\bar{S}_{\Delta}^{(q)}(x)\right| &\leq C_q h^{8-q}\omega_7(h)+\bar{T}_q h^{8-q}\omega_7(h)\\ &=(C_q+\bar{T}_q)\,h^{8-q}\omega_7(h) \qquad ,\\ &=U_q h^{8-q}\omega_7(h)\end{aligned}$$

where $q=0,1,....,8,$ Which is proves (ii).

Theorem 2: If the function f in problem (1) satisfies conditions (2) and (3), then the following inequalities are hold:

$\left|\bar{S}_0''(x)-f[x,\bar{S}_0(x),\bar{S}_0'(x)]\right|\leq T_{0,2}^{*}\,\omega_7(h)$ where $T_{0,2}^{*}$ is constants dependent of h and $x\in[x_0,\,x_1]$,

$\left|\bar{S}_i''(x)-f[x,\bar{S}_i(x),\bar{S}_i'(x)]\right|\leq T_{i,2}^{*}\,\omega_6(h)$ where $T_{i,2}^{*}$ is constants dependent of h and $x\in[x_{i-1},\,x_i]$,

$\left|\bar{S}_{m-1}''(x)-f[x,\bar{S}_{m-1}(x),\bar{S}_{m-1}'(x)]\right|\leq T_{m-1,2}^{*}\,\omega_6(h)$ where $T_{m-1,2}^{*}$ is constants dependent of h and $x\in[x_{m-1},\,x_m]$.

Proof: Using condition (1), (2) and (3), we have

$$\begin{aligned}\left|\bar{S}_\Delta''(x)-f[x,\bar{S}_\Delta(x),\bar{S}_\Delta'(x)]\right| &\leq\left|\bar{S}_\Delta''(x)+y''(x)-y''(x)-f[x,\bar{S}_\Delta(x),\bar{S}_\Delta'(x)]\right|\\ &\leq\left|\bar{S}_\Delta''(x)-y''(x)\right|+\left|y''(x)-f[x,\bar{S}_\Delta(x),\bar{S}_\Delta'(x)]\right|\\ &\leq\left|\bar{S}_\Delta''(x)-y''(x)\right|+L\left\{\left|\bar{S}_\Delta(x)-y(x)\right|+\left|\bar{S}_\Delta'(x)-y'(x)\right|\right\}\end{aligned}$$

That is proves theorem 2 with the help theorem 1.

Note: Similar manner to theorem 2 was proved under different conditions by Saxena [9], and Gyorvari [3].

4 Numerical Results

Finally, we proceed to show numerical tests of the described algorithms. We want to analyze the local regularity of the derivatives method; we now consider two numerical examples illustrating the comparative performance of spline method. All calculations are implemented by Matlab program [1-2]. For the sake of comparisons we also tabulated the results seen that the present method is better than method [5].

Algorithm:

Step 1: Partition $[a,b]$ into N subintervals I.

Step 2: Set

$\overline{S}_i' = \overline{y}_i'$ (i=0, 1, 2, …, N) , $\overline{S}_i = \overline{y}_i$ (i=0, 1, 2, …, N)

$\overline{S}_i^{(4)} = \overline{y}_i^{(4)}$ (i=0, 1, 2, …, N-1)

and with initial condition $\overline{S}_0''' = \overline{y}_0'''$.

Step 3: Use (Theorem 1 (i)) to find $S_i - \overline{S}_i$, $i = 1, 2, \ldots, N$.

Step 4: Use (Theorem 1 (i)) to find the derivatives of $S_i - \overline{S}_i$ at N equally spaced points in each subinterval $x \in [x_{i-1}, x_i]$ go to step 5, else i=i+1 and repeat this iteration to find a proper i.

Step 5: Stop.

Problem 1: [6] Consider that the fifth order intial value problem $y^{(5)} - y^{(4)} - y' + y = 0$ where $x \in [0,1]$, $y(0) = y'(0) = y''(0) = y'''(0) = y^{(4)}(0) = 1$, and the exact solution is $y(x) = e^x$.

Problem 2: [10] Consider that the third order initial value problem $y''' + 2y'' + y' + 2y = 0$ where $t \in [0,1]$, $y(0) = 3, y'(0) = -2$, $y''(0) = 3$ the exact solution is $y(x) = e^{-2t} + \cos(t) + 1$.

Table 1 Absolute maximum error for the derivatives $\overline{S}(x)$.

h	$\lVert \bar{s}''(x) - y''(x) \rVert_\infty$	$\lVert \bar{s}'''(x) - y'''(x) \rVert_\infty$	$\lVert \bar{s}^{(5)}(x) - y^{(5)}(x) \rVert_\infty$
0.1	8.4768×10^{-8}	4.2537×10^{-6}	5.2×10^{-3}
0.01	4.7657×10^{-11}	$2.38571 \times 10^{-8`}$	5.6×10^{-3}
0.001	1.3892×10^{-18}	2.0839×10^{-14}	8.3354×10^{-7}
0.0001	$1.38.96 \times 10^{-23}$	2.0844×10^{-18}	8.3378×10^{-7}

h	$\lVert \bar{s}^{(6)}(x) - y^{(6)}(x) \rVert_\infty$	$\lVert \bar{s}^{(7)}(x) - y^{(7)}(x) \rVert_\infty$
0.1	10.38×10^{-2}	10.45×10^{-1}
0.01	28.047×10^{-1}	505.68×10^{0}
0.001	2.5×10^{-3}	35.009×10^{-1}
0.0001	2.5013×10^{-4}	3.5019×10^{0}

Table 2 Absolute maximum error for the derivatives $\bar{S}(x)$.

h	$\left\|\bar{s}''(x)-y''(x)\right\|_\infty$	$\left\|\bar{s}'''(x)-y'''(x)\right\|_\infty$	$\left\|\bar{s}^{(5)}(x)-y^{(5)}(x)\right\|_\infty$
0.1	1.0314×10^{-5}	5.1209×10^{-4}	6.027×10^{-1}
0.01	6.6183×10^{-11}	3.3061×10^{-8}	1.12×10^{-2}
0.001	1.7769×10^{-16}	2.6653×10^{-12}	1.0661×10^{-4}
0.0001	1.7762×10^{-11}	6.661×10^{-7}	1.3323×10^{-3}

h	$\left\|\bar{s}^{(6)}(x)-y^{(6)}(x)\right\|_\infty$	$\left\|\bar{s}^{(7)}(x)-y^{(7)}(x)\right\|_\infty$
0.1	11.902×10^{0}	117.3×10^{0}
0.01	36.76×10^{-1}	559.01×10^{0}
0.001	3.198×10^{-1}	447.77×10^{0}
0.0001	6.3949×10^{0}	111.91×10^{0}

6 Conclusion

As we expected, the maximum absolute errors in the solution of the fifth and third order initial value problems given by our construction are smaller than the errors in the constructions in [5,7 and 8], also we can using this model to find the approximate solution for all order initial value problems with a good result even for small h. Moreover, we found new construction, gives more accurate results in comparison with sixth and seventh spline used in [7].

7 Open Problem

In this work, we present a numerical method for solving the higher order initial value problems; we can develop the idea for boundary value problems, system of differential equations and partial differential equations difference type of boundary conditions.

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