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On Differential Ideals of Differential rings

Yaseen A.W.Alhiti

Ishik University Erbil – Iraq

E-mail : al_hiti@yahoo.com

Abstract

In this paper we introduce two operators denoted by $(\ )_{(n)}$ and $(\ )_u$ of a differential ring constructed from a subset of a differential ring. We shall also discuss the relationship between these operators and the differential ideals in differential rings, and Keigher differential ring.

Keywords : Differential ring , Keigher differential ring , Prime differential spectrum.

1 Introduction

Rings considered in this paper are all commutative with unity. The 0 ring has $1=0$. Also, all differential rings are ordinary , i.e., posses a single derivation .Recall that by a derivation of a ring R we means any additive map $\delta : R \to R$ satisfying $\delta(ab) = \delta(a)b + a\delta(b)$ for every $a,b \in R$.A differential ring R is a ring with a derivation δ. If R is a differential ring and $a \in R$, then $a^{(n)}$ denotes the n th derivative of a. A subset A of R is called differential if $\delta(A) \subseteq A$.For any subset A of R , the set $A_{\delta} = \{a \in A : \delta(a) \in A\}$ is called the differential of A.

Let R be a differential ring and let A be a subset of R . We define a subset , denoted by $A_{(n)}$,of R by $A_{(n)} = \{a : a^{(n)} \in A,$ for all $n \geq 0\}$. The following two theorems give some of the properties of $A_{(n)}$.

Theorem 1.1. *Let R be a differential ring . Then*

(1) *If $A \subset R$, then $A_{(n)} \subset A$ and $(A_{(n)})_{(n)} = A_{(n)}$.*
(2) *If $A \subset R$, then $A_{(n)} = A$ iff A is differential subset of R .*
(3) *If A, B are subsets of R with $A \subset B$, then $A_{(n)} \subset B_{(n)}$.*
(4) *If $\{A_\alpha\}_{\alpha \in I}$ is a family of subsets of R , then $(\bigcap_{\alpha \in I} A_\alpha)_{(n)} = \bigcap_{\alpha \in I} (A_\alpha)_{(n)}$ and $(\bigcup_{\alpha \in I} A_\alpha)_{(n)} \supset \bigcup_{\alpha \in I} (A_\alpha)_{(n)}$.*
(5) *If A, B are subsets of R , then $(A+B)_{(n)} \supset A_{(n)} + B_{(n)}$ and $(A.B)_n \supset A_{(n)}.B_{(n)}$.*

Theorem 1.2. *Let R and S be differential rings and let $\varphi : R \to S$ be differential ring homomorphism such that $\varphi\,(1) = 1$.. If A is a subset of R and B is a subset of S , then $\varphi(A_{(n)}) = (\varphi(A))_{(n)}$ and $\varphi^{-1}(B_{(n)}) = (\varphi^{-1}(B))_{(n)}$.*

The proof of these theorems is elementary and follows immediately from the definitions .

From theorems 1.1 and 1.2, we see that for any subset A of a differential ring R , $A_{(n)}$ is a differential subset . Also, the union and the intersection of any family of differential subsets is again a differential subset , and finite sums and products of differential subsets are differential subsets . Moreover , direct and inverse images of differential subsets under a differential ring homomorphism are differential.

Let A be a subset of a differential ring R .We define a subset , denoted by A_u , of R by $A_u = \{a \in A : \exists b \in A$ such that $ab = 1\}$. Hence , if A is a subring of R , A_u is the set of units in A.

Theorem 1.3. *Let R be a differential ring and S a subring of R . Then $(S_{(n)})_u = S_{(n)} \cap S_u$.*

Proof. It is clear that $(S_{(n)})_u \subset S_{(n)} \cap S_u$, so let $a \in S$ be such that $a^{(n)} \in S$ for all $n \geq 0,$ and suppose that $ab = 1$ for some $b \in S$ .We want to show that $b^{(n)} \in S$ for all $n \geq 0$. We may assume $n \geq 1$ and that for each $k < n, b^{(k)} \in S$. Then by Leibnit$z$'s rule [6] we have

$$0 = (ab)^{(n)} = ab^{(n)} + \sum_{k=1}^{n} \binom{n}{k} a^{(k)} b^{(n-k)} \ ,$$

So that

$$b^{(n)} = -b\Big(\sum_{k=1}^{n} \binom{n}{k} a^{(k)} b^{(n-k)} \Big) \in S$$

Hence , $(S_{(n)})_u = S_{(n)} \cap S_u$.

2. DIFFERENTIAL IDEALS AND KEIGHER RINGS

Theorem 2.1. *Let R be a differential ring and let A be a subset of R , then*

(1) If A is a subring of R , then $A_{(n)}$ is a subring of R.
(2) If A is an ideal of R , then $A_{(n)}$ is an ideal of R.

Proof. The proof of part (1) follows immediately from the definition. To prove part (2) ,suppose $x \in R$ and $a \in A_{(n)}$. Then by Leibentiz's rule [6] we have

$$(x\, a)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} a^{(n-k)}$$

Since every $a^{(n-k)} \in A$ and A is an ideal of $R, (x\, a)^{(n)} \in A$ and hence $x\, a \in A_{(n)}$. So that , $A_{(n)}$ is an ideal of R.

Recall that by a Ritt algebra [5] we means any differential ring which contains the rational numbers. Also, if I is an ideal of a differential ring R, the set $r(I) = \{a \in R : a^n \in I$ for some $n \in \mathbb{Z}^+\}$ is called the radical of I . An ideal I of R is called a radical ideal if $r(I) = I$.

Theorem 2.2. *Let R be a Ritt algebra and let I be a subset of R, then*

(1) If I is a prime ideal of R , then $I_{(n)}$ is a prime ideal of R.
(2) If I is a radical ideal of R , then $I_{(n)}$ is a radical ideal of R.

Proof. (1) From theorem 1.4 we have $I_{(n)}$ is an ideal of R, so suppose that $a \notin I_{(n)}$ and $b \notin I_{(n)}$. Then there exist positive integers m, n such that $a^{(m)} \notin I$, $b^{(n)} \notin I$ and for all $k < m$ and $l < n$, $a^{(k)} \in I$ and $b^{(l)} \in I$. Now let

$$(ab)^{(m+n)} = \sum_{k=0}^{m+n} \binom{m+n}{k} a^{(k)} b^{(m+n-k)}$$

We note that , $\binom{m+n}{k} a^{(k)} b^{(m+n-k)} \in I$ for $k < m$, while for $k > m$, i.e. , for $m+n-k < n$, $\binom{m+n}{k} a^{(k)} b^{(m+n-k)} \in I$.
If $k = m$, $a^{(m)} b^{(n)} \notin I$ since I is a prime ideal , and since R is a Ritt algebra $\binom{m+n}{m} a^{(m)} b^{(n)} \notin I$. Hence $(ab)^{(m+n)} \notin I$, so that $I_{(n)}$ is a prime ideal.

(2) Note that every radical ideal of R is an intersection of prime ideals of R and conversely . Since the operator $(\ )_{(n)}$ preserves the intersection of ideals by Theorem 1.1 and prime ideals by part (1) , we have well that $(\ )_{(n)}$ preserves the radical ideals.

Definition 2.3 [7] . *Let R be a differential ring , R is said to be a Keigher ring if for each prime ideal I in R , $I_{(n)}$ is also prime ideal in R.*

Examples.

1. Every Ritt algebra R is a Keigher ring by the above Theorem 2.2.
2. Every differential field F is a keigher ring.
3. Every ring R with trivial derivation (i.e., $a^{(n)} = 0$ for all $a \in R$ and $n \geq 1$) is a Keigher ring.

Theorem 2.4. *Let R be a Keigher differential ring and $\varphi : R \to S$ a surjective differential ring homomorphism. Then S is also a Keigher ring.*

Proof. Since φ is surjective , then φ induces a one-to-one correspondence between prime ideals J in S and prime ideals I in R containing the kernel of φ via $I = \varphi^{-1}(J)$ and $J = \varphi(I)$. Hence if J is a prime ideal in S, then we have $J_{(n)} = \varphi(\varphi^{-1}(J_{(n)})) = \varphi((\varphi^{-1}(J))_{(n)})$. But since R is a Keigher ring , $(\varphi^{-1}(J))_{(n)}$ is prime ideal in R and hence $J_{(n)}$ is prime ideal in S.

Recall that if S is a multiplicative subset of a differential ring R , then the ring opf fractions $S^{-1}R$ is a differential ring via $(\frac{r}{s})^{(1)} = \frac{s\,r^{(1)} - r\,s^{(1)}}{s^{(2)}}$, see [2].

The following lemma was proved by Keigher in [7].

Lemma 2.5. *Let R be a differential ring . Let S be a multiplicative subset of R and I a prime ideal in R such that $I \cap S = \varnothing$. Then in the differential ring $S^{-1}R$ we have*
$(S^{-1}I)_{(n)} = S^{-1}I_{(n)}$.

Theorem 2.6. *Let R be a Keigher differential ring and S a multiplicative subset of R. Then $S^{-1}R$ is also a Keigher ring.*

Proof. The proof follows immediately from the Lemma 2.1 , since there is a one-to-one correspondence between prime ideals of $S^{-1}R$ and prime ideals of R disjoint from S [7].

Corollary 2.7. *Let R be a differential ring and let P be a prime ideal of R , then R is a Keigher differential ring if and only if R_P is a Keigher ring .*

Proof. If R is a Keigher ring , then so every R_P by Theorem 2.4. Conversely , let P be a prime ideal of R and let $f : R \to R_P$ be the canonical differential ring homomorphism . Let $S = R - P$, then since $P = f^{-1}(S^{-1}P)$, we see that $P_{(n)} = f^{-1}((S^{-1}P)_{(n)})$ by Theorem 1.1, and since R_P is a Keigher ring , $(S^{-1}P)_{(n)}$ is prime in R_P . Hence $P_{(n)}$ is prime in R and R is a Keigher ring.

Theorem 2.8. *Let $R = \prod_{i=1}^{n} R_i$, where R_i is differential ring . Then R is a Keigher ring if and only if each R_i is a Keigher ring.*

Proof. If R is a Keigher ring , then so is each R_i by Theorem 2.3 . Conversely suppose that I is a prime ideal of R , and let $\pi_i : R \to R_i$, $i = 1, 2, ..., n$, be the canonical projections. Then $\pi_k(I) = I_k$, $1 \le k \le n$,is a prime ideal in R_k and $\pi_j(I) = R_j$ for $j \ne k$. It is clear that $I_{(n)} = \pi_k^{-1}((I_k)_{(n)})$, and since R_k is a Keigher ring , $I_{(n)}$ is prime ideal of R and R is a Keigher ring.

Definition 2.9 [5] . A differential ring R is called a $d-MP$ ring if the radical of a differential ideal I of R is again a differential ideal. This is equivalent , see [2] , [3], [8], to each of the following :

(1) Prime ideals minimal over differential ideals are differential ideals .

(2) If I is a differential ideal of R and S is a multiplicative subset of R disjoint from I , then ideals maximal among differential ideals which contain I and are disjoint from S are prime.

Theorem 2.10. *Let R be a differential ring . Then R is a Keigher ring if and only if it is a $d-MP$ ring .*

Proof. See [7] .

Let R be a differential ring . A differential ideal I is prime if and only if there is a multiplicative subset S of R such that I is maximal among ideals disjoint from S [6].

Let R be a differential ring . A differential ideal I is called quasi- prime ideal if there is a multiplicative subset S of R such that I is maximal among differential ideals disjoint from S . It is clear that every prime differential ideal is quasi-prime, and every quasi-prime ideal is prime if and only if R is a Keigher ring.

Theorem 2.11. *Let R be a differential ring . If I is a prime ideal of R then $I_{(n)}$ is a quasi-prime.*

Proof . Let I be a prime ideal of R and let $S=R-I$. It is clear that $I_{(n)}$ is a differential ideal disjoint from S and if J is any differential ideal disjoint from S , then $J \subset I$, so that $J=J_{(n)} \subset I_{(n)}$. Hence $I_{(n)}$ is maximal among differential ideals disjoint from S . Now let K be a quasi- prime ideal of R and let S be a multiplicative subset of R such that K is maximal among differential ideals disjoint from S . Then there exists a prime ideal I of R such that $K \subset I$ and $I \cap S=\varnothing$ [1] .Hence $K=K_{(u)} \subset I_{(u)}$ and $I_{(n)} \cap S=\varnothing$, so that $K=I_{(n)}$.

3 The Prime Spectrum of a differential ring

In the sense of ring theory , for any commutative ring R , $\operatorname{Spec}(R)$ denote the set of prime ideals in R with the Zariski topology [4]. The following two theorems show how to create a topological space from a commutative ring R .

This topological space is called the prime spectrum of R and the topology is called the Zariski topology.

Theorem 3.1. *Let R be a commutative ring and let $Spec(R)$ be the set of all prime ideals of R. For any subset A of R let $V(A)$ be the set of all prime ideals of R that contain A. Then*
(1) $V(A) = V((A))$ for any subset A of R (where (A) is the ideal generated by A).
(2) $V(0) =$ Spec (R) and $V(R) = \varnothing$.
(3) If $\{A_i\}_{i\in I}$ is a family of subsets of R, then $V(\bigcup_{i\in I} A_i) = \bigcap_{i\in I} V(A_i)$.
(4) If A and B are two subsets of R, then $V(A \cap B) = V(A) \cup V(B)$.

Parts $(2),(3)$ and (4) show that the sets $V(A)$, as A runs over all subsets of R, satisfy the axioms for a collection of closed sets in a topological space . The subset $V(A)$ of Spec(R) are called Zarisky closed sets . Henceforth , Spec(R) is considered to have the topology defined by taking the Zariski closed sets to be the closed sets $-$ this is the Zariski topology on Spec(R).

Theorem 3.2. *Let R and S be commutative rings and let $\varphi : R \to S$ be a ring homomorphism such that $\varphi(1) = 1$.*
(1) If I is a prime ideal of S, then $\varphi^{-1}(I)$ is a prime ideal of R. Thus φ induce a map

$\varphi^* : Spec(S) \to Spec(R)$ defined by $\varphi^*(I) = \varphi^{-1}(I)$ for all $I \in Spec(S)$.

(2) For any ideal J in R, ${\varphi^*}^{-1}(V(J)) = V((\varphi(J)))$ (where $(\varphi(J))$ is the ideal generated by $\varphi(J)$ in S). Deduce that φ^* is a continuous map with respect to the Zariski topology on $Spec(S)$ and $Spec(R)$.
(3) If $\Omega : S \to T$ is also a homomorphism of commutative rings , then $(\Omega \circ \varphi)^* = \varphi^* \circ \Omega^*$.

Proof . The proof follows directly from the definitions , see [4].

If R is a differential ring , the set of prime differential ideals in R will be denoted by Spec$_d(R)$ and will be called the prime differential spectrum of R. As a topologi- cal space , the set Spec$_d(R)$ has the subspace topology from

$\operatorname{Spec}(R)$. So that the closed sets in $\operatorname{Spec}_d(R)$ are defined by the form $V_*(A) = V(A) \cap \operatorname{Spec}_d(R)$, where A is a subset of R.

Denote by $r_d(I)$ the differential radical of differential ideal I of R and I is called a differential radical ideal if $I = r_d(I)$.

For an element $a \in R$ denote by $[a]$ the smallest differential ideal containing a.

Some of properties of differential radical ideals are given in the following theorems.

Theorem 3.3 [8] . *For a differential ring R the following conditions are equivalent :*

(1) *Every differential ideal of R is differential radical ideal .*

(2) $I.J = I \cap J$ *for all differential ideals I, J in R .*

(3) $[a]^2 = [a]$ *for all* $a \in R$.

If $r((A))$ denotes the radical of the ideal in R generated by A , $r_d(A)$ denotes the differential radical of A , and $r_d(A)$ can be defined as following :

Theorem 3.3[8] . *For any subset A of a differential ring R, the differential radical of A , $r_d(A)$ is the intersection of all differential prime ideals in R containing A .*

It is clear that , $A \subset r((A)) \subset r_d(A)$ and $r_d(r_d(A)) = r_d(A)$, where A is a subset of R. If Y is a subset of $\operatorname{Spec}_d(R)$, let $V_d(Y)$ denote the intersection of all prime differential ideals of R which belong to Y . It is easy to show that [9] :

(1) $V_d(Y)$ is a differential ideal of R, and the map from $\operatorname{Spec}_d(R)$ to R given by $Y \mapsto V_d(Y)$ is order – reversing with respect to the partial ordering by inclusion in $\operatorname{Spec}_d(R)$ and R.

(2) $V_d(\varnothing) = R$.

(3) If $\{Y_i\}_{i \in I}$ is a family of subsets of $\operatorname{Spec}_d(R)$, then $V_d(\bigcup_{i \in I} Y_i) = \bigcap_{i \in I} V_d(Y_i)$.

Theorem 3.4. *Let R be a differential ring , A a subset of R , and Y a subset of*

$Spec_d(R)$. Then

(1) $V_*(A)$ is closed in $Spec_d(R)$ and $V_d(Y)$ is a differential radical ideal of R.

(2) $V_d(V_*(A))$ is the differential radical of A and $V_*(V_d(Y))$ is the closure of Y in $Spec_d(R)$.

Proof. The proof follows from the definitions and the notes above.

Now let R and S be differential rings and $\psi: R \to S$ be a differential ring homo-
morphism . Then ψ induce a continuous map $\psi^*: \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ given by
$\psi^*(P) = \psi^{-1}(P)$ for all $P \in \mathrm{Spec}(S)$. It follows from Theorems 1.2 , 3.2 that ψ^* restricts to give a continuous map ψ_d^* : $\mathrm{Spec}_d(S) \to \mathrm{Spec}_d(R)$. If $\phi: S \to T$ is another differential ring homomorphism , then it is clearly that $(\phi \circ \psi)_d^* = \psi_d^* \circ \phi_d^*$.

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