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Explicit Invariant Regions and Global Existence of Solutions for Reaction Diffusion Systems With a Full Matrix of Diffusion Coefficients and Nonhomogeneous Boundary Conditions.

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Abstract

In this paper we construct invariant regions only in terms of the eigenvalues and entries of the diffusion matrix associated to a class of reaction diffusion systems with a general full matrix of diffusion coefficients and nonhomogeneous boundary conditions. This generalizes all the above papers (see S. Abdelmalek [1] in the case of tripled component systems and S. Kouachi [15], [16], [17] and [18]) where invariant regions are constructed in terms of a very complicated constants. In these regions we establish global existence of solutions for reaction-diffusion systems without balance law-condition ($f + g \equiv 0$). Our techniques are based on invariant regions and Lyapunov functional methods. The nonlinear reaction terms has been supposed to be of polynomial growth.

Keywords: Reaction diffusion systems, Invariant regions, Lyapunov functionals, Global existence.

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1 Introduction

We consider the following reaction-diffusion system

$$\frac{\partial u}{\partial t} - a_{11}\Delta u - a_{12}\Delta v = f(u,v) \text{ in } \mathbb{R}^+ \times \Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - a_{21}\Delta u - a_{22}\Delta v = g(u,v) \text{ in } \mathbb{R}^+ \times \Omega, \tag{1.2}$$

with the boundary conditions

$$\lambda u + (1-\lambda)\frac{\partial u}{\partial \eta} = \beta_1 \text{ and } \lambda v + (1-\lambda)\frac{\partial v}{\partial \eta} = \beta_2 \text{ on } \mathbb{R}^+ \times \partial\Omega, \tag{1.3}$$

where Robin nonhomogeneous boundary conditions ($0 < \lambda < 1$ and $\beta_i \in \mathbb{R},\ i = 1$ and 2) or homogeneous Neumann boundary conditions ($\lambda = \beta_i = 0,\ i = 1$ and 2) or homogeneous Dirichlet boundary conditions ($1 - \lambda = \beta_i = 0,\ i = 1$ and 2.) are assumed and the initial data

$$u(0,x) = u_0(x), \qquad v(0,x) = v_0(x) \text{ in } \Omega, \tag{1.4}$$

where Ω is an open bounded domain of class $\mathbb{C}^1$ in $\mathbb{R}^N$, with boundary $\partial\Omega$ and $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$. The constants $a_{ij}, (i,j = 1,2)$ are supposed to be positive and satisfy

$$\left(a_{12} + a_{21}\right)^2 \leq 4a_{11}a_{22}\right) \tag{1.5}$$

which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is positive definite; that is the eigenvalues λ_1 and λ_2 ($\lambda_1 < \lambda_2$) of it's transposed are positive. The initial data and (β_1, β_2) are assumed to be in the following region

$$\Sigma = \begin{cases} \{(u_0,v_0) \in \mathbb{R}^2 \text{ such that } \lambda_1 v_0 \leq a_{21}u_0 + a_{22}v_0 \leq \lambda_2 v_0\}\,, \\ \text{or} \\ \{(u_0,v_0) \in \mathbb{R}^2 \text{ such that } \lambda_1 u_0 \leq a_{11}u_0 + a_{12}v_0 \leq \lambda_2 u_0\}\,. \end{cases} \tag{1.6}$$

One will treat the first case, the second one will be discussed at the last section. We suppose that the reaction terms f and g are continuously differentiable, polynomially bounded on Σ, $(f(r,s), g(r,s))$ is in Σ for all (r,s) in $\partial\Sigma$ (we say that (f,g) points into Σ on $\partial\Sigma$); that is

$$\begin{cases} \lambda_1 g(r,s) \leq a_{21}f(r,s) + a_{22}g(r,s), \text{ for all } r \text{ and } s \text{ such that } \lambda_1 s = a_{21}r + a_{22}s \\ \text{and} \\ a_{21}f(r,s) + a_{22}g(r,s) \leq \lambda_2 g(r,s), \text{ for all } r \text{ and } s \text{ such that } a_{21}r + a_{22}s = \lambda_2 s, \end{cases} \tag{1.7}$$

and for positive constants C and $\alpha > a_{22} - \lambda_1$sufficiently close to $a_{22} - \lambda_1$, we have

$$a_{21}f(u,v) + Cg(u,v) \leq C_1 \left(a_{21}u + \alpha v + 1\right) \text{ for all } u \text{ and } v \text{ in } \Sigma \tag{1.8}$$

where C_1 is a positive constant.

The trivial case where $a_{12} = a_{21} = a_{11} - a_{22} = 0$; nonnegative solutions exist globally in time. Always in this case with homogeneous Neumann boundary conditions but when $a_{11} \neq a_{22}$ (diagonal case), N. Alikakos [2] established global existence and L^∞-bounds of solutions for positive initial data when

$$g(u,v) = -f(u,v) = uv^\beta, \tag{1.9}$$

and $1 < \beta < \frac{(n+2)}{n}$. The reactions given by (1.8) satisfy in fact a condition analogous to (1.7) and form a special case since (f,g) point into Σ on $\partial\Sigma$ by taking $\Sigma = \mathbb{R}^+ \times \mathbb{R}^+$. K. Masuda [21] showed that solutions to this system exist globally for every $\beta > 1$ and converge to a constant vector as $t \to +\infty$. A. Haraux and A. Youkana [6] have generalized the method of K. Masuda to handle nonlinearities $uF(v)$ that are form a particular case of our one; since they took also $\Sigma = \mathbb{R}^+ \times \mathbb{R}^+$. Recently S. Kouachi and A. Youkana [19] have generalized the method of A. Haraux and A. Youkana to the triangular case $(a_{12} = 0)$ and by taking nonlinearities $f(u,v)$ of a weak exponential growth. J. I. Kanel and M. Kirane [10] have proved global existence , in the case $g(u,v) = - \ f(u,v) = uv^n$ and n is an odd integer, under the embarrassing condition

$$|a_{12} - a_{21}| < C_p, \tag{1.10}$$

where C_p contains a constant from an estimate of Solonnikov. Then they ameliorate their results in [11] to obtain global existence under the restrictive conditions

$$\begin{cases} a_{22} < a_{11} + a_{21}, \\ a_{12} < \varepsilon_0 \equiv \left(\frac{a_{11}a_{22}(a_{11}+a_{21}-a_{22})}{a_{11}a_{22}+(a_{11}+a_{21}-a_{22})}\right) \text{ if } a_{11} \leq a_{22} < a_{11} + a_{21}, \\ a_{12} < \min\left\{\frac{1}{2}\left(a_{11}+a_{21}\right), \varepsilon_0\right\}, \end{cases} \tag{1.11}$$

and

$$|F(v)| \leq C_F(1 + |v|^{1+\varepsilon}, \tag{1.12}$$

where ε and C_F are positive constants with $\varepsilon < 1$ sufficiently small and $g(u,v) = - \ f(u,v) = uF(v)$. All techniques used by authors cited above showed their limitations because some are based on the embedding theorem of Sobolev as Alikakos [1], Hollis-Martin-Pierre [8], ... another as Kanel-Kirane [11] used a properties of the Neumann function for the heat equation for which one of it's restriction the coefficient of $-\Delta u$ in equation (1.1) must be bigger than the one of $-\Delta v$ in equation (1.2) whereas it isn't the case of problem (1.1)-(1.4).

This article is a continuation of [16] where $a_{11} = a_{22}$ and $\sigma g + \rho f \equiv 0$ with σ and ρ are any positive constants and the function $g(u, v)$ is positive and polynomially bounded. In that article we have considered the homogeneous Neumann boundary conditions and established global existence of solutions with initial data in an invariant region which is a special case of that considered here. Recently in S. Kouachi [18] and always in the case where $a_{11} = a_{22}$, we have eliminated the balance's condition which has been replaced by a condition analogous to (1.8).

The components $u(t, x)$ and $v(t, x)$ represent either chemical concentrations or biological population densities and system (1.1)-(1.2) is a mathematical model describing various chemical and biological phenomena (see P. L. Garcia-Ybarra and P. Clavin [4], S. R. De Groot and P. Mazur [5], J. Jorne [9], J. S. Kirkaldy [14], A. I. Lee and J. M. Hill [20] and J. Savchik, B. Changs and H. Rabitz [23].

2 Local existence and Invariant regions.

In this section, we prove that if (f, g) points into Σ on $\partial\Sigma$ then Σ is an invariant region for problem (1.1)-(1.4), i.e. the solution $(u(t, .), v(t, .))$ remains in Σ for all $(u_0, v_0) \in \Sigma$, u_0, v_0 bounded in Ω . At this stage and once the invariant regions are constructed, both problems of local and global existence become easier to be established: For the first problem we demonstrate that system (1.1)-(1.2) with boundary conditions (1.3) and $(u_0, v_0) \in \Sigma$, u_0, v_0 bounded in Ω is equivalent to a problem for which local existence over the whole time interval $[0, T_{\max}[$ can be obtained by known procedure and for the second, since we use usual techniques based on Lyapunov functionals which are not directly applicable to problem (1.1)-(1.4) and need invariant regions (see M. Kirane and S. Kouachi [12], [13], S. Kouachi [15] and [16] and S. Kouachi and A. Youkana [19]).

The main result of this section is the following

Suppose that (f, g) points into Σ on $\partial\Sigma$, then for any (u_0, v_0) in Σ the solution $(u(t, .), v(t, .))$ of the problem (1.1)-(1.4) remains in Σ for all t in $[0, T_{\max}[$.

Let $T^* < T_{\max}$ be an arbitrary positive number and let λ_1 and λ_2 $(\lambda_1 < \lambda_2)$ the eigenvalues of the matrix A^t associated respectively with the eigenvectors $\begin{pmatrix} x_{11} & x_{12} \end{pmatrix}^t$ and $\begin{pmatrix} x_{21} & x_{22} \end{pmatrix}^t$. For $i = 1, 2$ fixed, multiplying equation (1.1) through by x_{i1} and equation (1.2) by x_{i2} and adding the resulting equations we get

$$\frac{\partial w_1}{\partial t} - \lambda_1 \Delta w_1 = (x_{11} f + x_{12} g) = F_1(w_1, w_2) \qquad \text{in }]0, T^*[\times \Omega \tag{2.1}$$

$$\frac{\partial w_2}{\partial t} - \lambda_2 \Delta w_2 = (x_{21} f + x_{22} g) = F_2(w_1, w_2) \qquad \text{in }]0, T^*[\times \Omega, \tag{2.2}$$

with the boundary conditions

$$\lambda w_i + (1-\lambda)\frac{\partial w_i}{\partial \eta} = \rho_i, \ i = 1, 2 \qquad \text{on }]0, T^*[\times \partial\Omega\ , \tag{2.3}$$

and the initial data

$$w_i(0, x) = w_i^0(x), \ i = 1, 2 \qquad \text{in } \Omega, \tag{2.4}$$

where

$$w_i = (x_{i1} u + x_{i2} v)(t, x), \ i = 1, 2 \qquad \text{in }]0, T^*[\times \Omega \tag{2.5}$$

for all (t, x) in $]0, T^*[\times \Omega$,

$$\rho_i = (x_{i1}\beta_1 + x_{i2}\beta_2)\ , \ i = 1, 2$$

and

$$F_i(w_1, w_2) = (x_{i1} f + x_{i2} g)\ , \ i = 1, 2, \ \text{ for all } u \text{ and } v \text{ in } \Sigma. \tag{2.6}$$

Remark that the condition (1.5) of parabolicity of the system (1.1)-(1.2) implies the one of the system (2.1)-(2.2) ; since it implies the positivity of the determinant of A which together with the positivity of its entries gives the eigenvalues λ_1 and λ_2 ($\lambda_1 < \lambda_2$) of the matrix A^t are positive. In these conditions we can conclude that problem (2.1)-(2.4) with diffusion coefficients λ_1 and λ_2 is equivalent to problem (1.1)-(1.4) and to prove that Σ given by (1.6) is an invariant region for system (1.1)(1.2) it suffices to prove that the region

$$\left\{(w_1^0, w_2^0) \in \mathbb{R}^2 \text{ such that } w_i^0 \geq 0, \, i = 1, 2, \right\} = \mathbb{R}^+ \times \mathbb{R}^+, \tag{2.7}$$

is invariant for system (2.1)-(2.2) and that

$$\Sigma = \left\{(u_0, v_0) \in \mathbb{R}^2 \text{ such that } w_i^0 = (x_{i1} u_0 + x_{i2} v_0) \geq 0, \, i = 1, 2, \right\}. \tag{2.8}$$

Since$\begin{pmatrix} x_i & x_{2i} \end{pmatrix}^t$ is an eigenvector of A^t associated to the eigenvalue λ_i, $i = 1, 2$, then if we assume without loss of generality that $a_{11} \leq a_{22}$ we have $(a_{11} - \lambda_i) x_{i1} + a_{21} x_{i2} = 0$, $i = 1, 2$. If we choose $x_{12} = a_{11} - \lambda_1$ and $x_{22} = \lambda_2 - a_{11}$ then $(x_{i1} u_0 + x_{i2} v_0) \geq 0$, $i = 1, 2, \Leftrightarrow -a_{21} u_0 + (a_{11} - \lambda_1) v_0 \geq 0$ and $a_{21} u_0 +$

$(\lambda_2 - a_{11})\, v_0 \geq 0 \Leftrightarrow -a_{21}u_0 + (\lambda_2 - a_{22})\, v_0 \geq 0$ and $a_{21}u_0 + (a_{22} - \lambda_1)\, v_0 \geq 0$. Then (2.8) is proved and (2.5) can be written

$$w_1 = -a_{21}u + (\lambda_2 - a_{22})\, v \text{ and } w_2 = a_{21}u + (a_{22} - \lambda_1)\, v.$$

Now, to prove that the region $\mathbb{R}^+ \times \mathbb{R}^+$ is invariant for system (2.1)-(2.2), it suffices to show that $F_1(w_1, w_2) \geq 0$ for all (w_1, w_2) such that $w_1 = 0$ and $w_2 \geq 0$ and $F_2(w_1, w_2) \geq 0$ for all (w_1, w_2) such that $w_1 \geq 0$ and $w_2 = 0$ thanks to the invariant region's method (see Smoller [24]). But using the expressions (2.7), we get

$$F_1(w_1, w_2) = -a_{21}f + (\lambda_2 - a_{22})\, g \text{ and } F_2(w_1, w_2) = a_{21}f + (a_{22} - \lambda_1)\, g. \quad (2.6)'$$

Following the same reasoning as above and taking in the account that $v_0 \geq 0$ in Σ, we lead to condition (1.7). Since $T^* < T_{\max}$ is arbitrary, then Σ is an invariant region for the system (1.1)-(1.3)

Then system (1.1)-(1.2) with boundary conditions (1.3) and initial data in Σ is equivalent to system (2.1)-(2.2) with boundary conditions (2.3) and positive initial data (2.4). As it has been mentioned at the beginning of this section and since ρ_1 and ρ_2 given by

$$\rho_1 = -a_{21}\beta_1 + (\lambda_2 - a_{22})\, \beta_2 \text{ and } \rho_2 = a_{21}\beta_1 + (a_{22} - \lambda_1)\, \beta_2$$

are positive, then for any initial data in $\mathbb{C}\left(\overline{\Omega}\right)$ or $\mathbb{L}^p(\Omega)$, $p \in (1, +\infty)$; local existence and uniqueness of solutions to the initial value problem (2.1)-(2.4) and consequently those of problem (1.1)-(1.4) follow from the basic existence theory for abstract semi-linear differential equations (see A. Friedman [3], D. Henry [7] and Pazy [22]). The solutions are classical on $]0, T^*[$, where T^* denotes the eventual blowing-up time in $\mathbb{L}^\infty(\Omega)$. The local solution is continued globally by a priori estimates.

The positivity of the matrix's diffusion's coefficients implies that

$$\lambda_1 < a_{11} < a_{22} < \lambda_2.$$

Once invariant regions are constructed, one can apply Lyapunov technique and establish global existence of unique solutions for (1.1)-(1.4).

3 Global existence.

As the determinant of the linear algebraic system (2.5), with regard to variables u and v, is different from zero, then to prove global existence of solutions of problem (1.1)-(1.4) comes back in even to prove it for problem (2.1)-(2.4). To this subject, it is well known that (see Henry [7]) it suffices to derive an uniform estimate of $\|F_1(w_1,w_2)\|_p$ and $\|F_2(w_1,w_2)\|_p$ on $[0,T^*[$ for some $p > N/2$, where $\|.\|_p$denotes the usual norms in spaces $\mathbb{L}^p(\Omega)$ defined by

$$\|u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p \, dx,\ 1 \leq p < \infty \text{ and } \|u\|_\infty = \underset{x\in\Omega}{ess\sup} |u(x)|$$

Let us define, for all positive integer p, the finite sequence

$$\theta_i = \theta^{(p-i)^2},\ i = 0,1,...,p \tag{3.1}$$

where θ is a positive constant sufficiently large such that

$$\theta > \frac{TrA}{2\sqrt{\det A}} \equiv \frac{(a_{11}+a_{22})}{2\sqrt{a_{11}a_{22}-a_{12}a_{21}}}. \tag{3.2}$$

Let us define, for a fixed positive integer p, the functional

$$t \longrightarrow L(t) = \int_\Omega H_p\left(w_1(t,x),w_2(t,x)\right)dx, \tag{3.3}$$

where

$$H_p(w_1,w_2) = \sum_{i=0}^{p} C_p^i \theta_i w_1^i w_2^{p-i}\ , \tag{3.4}$$

with C_p^i denotes the well known binomial coefficient. The main result of this section is the following

Let $(w_1(t,.),w_2(t,.))$be any positive solution of the problem (2.1)-(2.4), then the functional L defined by (3.3)-(3.4) is uniformly bounded on the interval $[0,T^*], T^* < T_{\max}$.

The proof is similar to that in S. Kouachi [15].

Differentiating L with respect to t yields

$$\begin{aligned}
L'(t) &= \int_\Omega \left[\tfrac{\partial H_p}{\partial w_1}\tfrac{\partial w_1}{\partial t} + \tfrac{\partial H_p}{\partial w_2}\tfrac{\partial w_2}{\partial t}\right]dx \\
&= \int_\Omega \left(\lambda_1 \tfrac{\partial H_p}{\partial w_1}\Delta w_1 + \lambda_2 \tfrac{\partial H_p}{\partial w_2}\Delta w_2\right)dx + \int_\Omega \left(F_1\tfrac{\partial H_p}{\partial w_1} + F_2\tfrac{\partial H_p}{\partial w_2}\right)dx \\
&= I + J.
\end{aligned}$$

By simple use of Green's formula we have

$$I = I_1 + I_2$$

where

$$I_1 = \int\limits_{\partial\Omega} \left(\lambda_1 \tfrac{\partial H_p}{\partial w_1} \tfrac{\partial w_1}{\partial \eta} + \lambda_2 \tfrac{\partial H_p}{\partial w_2} \tfrac{\partial w_2}{\partial \eta}\right) dx \tag{3.5}$$

and

$$I_2 = -\int\limits_{\Omega} \left(\lambda_1 \tfrac{\partial^2 H_p}{\partial w_1^2} \left|\nabla w_1\right|^2 + (\lambda_1+\lambda_2)\tfrac{\partial^2 H_p}{\partial w_1 \partial w_2} \nabla w_1 \nabla w_2 + \lambda_2 \tfrac{\partial^2 H_p}{\partial w_2^2} \left|\nabla w_2\right|^2\right) dx. \tag{3.6}$$

First, let's calculate the first and second partial derivatives of H_p with respect to w_1 and w_2. We have

$$\frac{\partial H_p}{\partial w_1} = \sum_{i=1}^{p} \left(i C_p^i \theta_i w_1^{i-1} w_2^{p-i}\right)$$

and

$$\frac{\partial H_p}{\partial w_2} = \sum_{i=0}^{p-1} \left((p-i) C_p^i \theta_i w_1^{i} w_2^{p-i-1}\right).$$

Using the formula

$$i C_p^i = p C_{p-1}^{i-1}, \text{ for all } \; i = 1, ..., p \tag{3.7}$$

and changing the index i by $i-1$, we get

$$\frac{\partial H_p}{\partial w_1} = p\sum_{i=0}^{p-1} \left(C_{p-1}^i \theta_{i+1} w_1^{i} w_2^{p-1-i}\right). \tag{3.8}$$

For $\dfrac{\partial H_p}{\partial w_2}$, using (3.7) and the fact that

$$C_p^i = C_p^{p-i}, \text{ for all } \; i = 0, ..., p, \tag{3.9}$$

we get

$$\frac{\partial H_p}{\partial w_2} = p\sum_{i=0}^{p-1} \left(C_{p-1}^i \theta_{i} w_1^{i} w_2^{p-1-i}\right). \tag{3.10}$$

Using formulas (3.8) and (3.10), we deduce by analogy

$$\frac{\partial^2 H_p}{\partial w_1^2} = p(p-1)\sum_{i=0}^{p-2} \left(C_{p-2}^i \theta_{i+2} w_1^{i} w_2^{p-2-i}\right), \tag{3.11}$$

$$\frac{\partial^2 H_p}{\partial w_1 \partial w_2} = p(p-1)\sum_{i=0}^{p-2} \left(C_{p-2}^i \theta_{i+1} w_1^{i} w_2^{p-2-i}\right) \tag{3.12}$$

and

$$\frac{\partial^2 H_p}{\partial w_2^2} = p(p-1)\sum_{i=0}^{p-2} \left(C_{p-2}^{i}\theta_i w_1^i w_2^{p-2-i}\right). \tag{3.13}$$

Now we claim that there exists a positive constant C_2 independent of $t \in [0, T_{\max}[$ such that

$$I_1 \leq C_2 \text{ for all } t \in [0, T_{\max}[. \tag{3.14}$$

To see this, we follow the same reasoning as in S. Kouachi [15]:
In the case of Robin nonhomogeneous boundary conditions, using the boundary conditions (1.4) we get

$$I_1 = \int_{\partial\Omega} \left(\lambda_1 \tfrac{\partial H_p}{\partial w_1}(\gamma_1 - \sigma w_1) + \lambda_2 \tfrac{\partial H_p}{\partial w_2}(\gamma_2 - \sigma w_2)\right) dx,$$

where $\sigma = \dfrac{\lambda}{1-\lambda}$ and $\gamma_i = \dfrac{\rho_i}{1-\lambda}$, $i = 1,\ 2.$
Since

$$\begin{aligned} H(w_1, w_2) &= a\tfrac{\partial H_p}{\partial u}(\gamma_1 - \sigma w_1) + b\tfrac{\partial H_p}{\partial v}(\gamma_2 - \sigma w_2) \\ &= P_{p-1}(w_1, w_2) - Q_p(w_1, w_2), \end{aligned}$$

where P_{p-1} and Q_p are polynomials with positive coefficients and respective degrees $p-1$ and p and since the solution is positive, then

$$\limsup_{(|w_1|+|w_2|)\to+\infty} H(w_1, w_2) = -\infty, \tag{3.15}$$

which prove that H is uniformly bounded on $\mathbb{R}_+^2$ and consequently (3.14).
When the boundary conditions are homogeneous of Neumann type, then $I_1 = 0$ on $[0, T_{\max}[$.
Finally the case of homogeneous Dirichlet conditions is trivial, since in this case the positivity of the solution on $[0, T_{\max}[\times\Omega$ implies $\dfrac{\partial w_1}{\partial\eta} \leq 0$ and $\dfrac{\partial w_2}{\partial\eta} \leq 0$ on $[0, T_{\max}[\times\partial\Omega$. Consequently one gets again (3.14) with $C_2 = 0$.

$$I_2 = -p(p-1)\sum_{i=0}^{p-2} C_{p-2}^{i} \int_{\Omega} w_1^i w_2^{p-2-i} T_i(\nabla w_1, \nabla w_2)\, dx,$$

where

$$\begin{aligned} T_i(\nabla w_1, \nabla w_2) &= \left(\lambda_1\theta_{i+2}|\nabla w_1|^2 + (\lambda_1+\lambda_2)\theta_{i+1}\nabla w_1 \nabla w_2 + \lambda_2\theta_i |\nabla w_2|^2\right), \\ i &= 0, 1, ..., p-2. \end{aligned}$$

Using (3.1) and (3.2) we deduce that the quadratic forms (with respect to ∇w_1 and ∇w_2) are positive since

$$\left(\left(\lambda_1+\lambda_2\right)\theta_{i+1}\right)^2-4\lambda_1\lambda_2\theta_i\theta_{i+2}=\theta_{i+1}^2\left(\left(\lambda_1+\lambda_2\right)^2-4\lambda_1\lambda_2\theta^2\right)<0,\ i=0,1,...,p-2. \tag{3.16}$$

Then

$$I_2\leq 0. \tag{3.17}$$

(3.8) and (3.10) together imply

$$J=p\sum_{i=0}^{p-1}C_{p-1}^{i}w_1^iw_2^{p-1-i}\int_\Omega G_i(w_1,w_2)dx,$$

where

$$G_i(w_1,w_2)=\left(\theta_{i+1}F_1(w_1,w_2)+\theta_iF_2(w_1,w_2)\right).$$

We have

$$\begin{aligned}G_i(w_1,w_2) &= a_{21}\left(-\theta_{i+1}+\theta_i\right)f(u,v)+\left[\left(\lambda_2-a_{22}\right)\theta_{i+1}+\left(a_{22}-\lambda_1\right)\theta_i\right]g(u,v)\\ &= \left[\left(\lambda_2-a_{22}\right)\theta_{i+1}+\left(a_{22}-\lambda_1\right)\theta_i\right]\left[a_{21}\Gamma\left(\frac{\theta_i}{\theta_{i+1}}\right)f(u,v)+g(u,v)\right],\end{aligned}$$

where

$$\Gamma\left(\frac{\theta_i}{\theta_{i+1}}\right)=\frac{\frac{\theta_i}{\theta_{i+1}}-1}{\left[(a_{22}-\lambda_1)\frac{\theta_i}{\theta_{i+1}}+(\lambda_2-a_{22})\right]},\ i=0,1,...,p-2.$$

Since the function $x\to\frac{x-1}{(a_{22}-\lambda_1)x+(\lambda_2-a_{22})}$ is increasing with $\lim\limits_{x\to+\infty}\frac{x-1}{(a_{22}-\lambda_1)x+(\lambda_2-a_{22})}=\frac{1}{(a_{22}-\lambda_1)}$ and since $\frac{\theta_i}{\theta_{i+1}}$ is sufficiently large by choosing θ sufficiently large, then by using condition (1.8) and relation (2.6) successively we get, for an appropriate constant C_3,

$$J\leq C_3\int_\Omega\left[\sum_{i=1}^{p}\left(w_1+w_2+1\right)C_{p-1}^{i-1}w_1^{i-1}w_2^{p-i}\right]dx.$$

Following the same reasoning as in S. Kouachi [17], a straightforward calculation shows that

$$L'(t)\leq C_4L(t)+C_5L^{(p-1)/p}(t)\text{ on }[0,T^*].$$

While putting

$$Z=L^{1/p},$$

one gets

$$pZ'\leq C_4Z+C_5.$$

The resolution of this linear differential inequality yields the uniform boundlessness of the functional L on the interval $[0, T^*]$, what finishes at the same time our reasoning and the proof of the theorem.

Suppose that the functions $f(r,s)$ and $g(r,s)$ are continuously differentiable on Σ, point into Σ on $\partial\Sigma$ and satisfy condition (1.8). Then all solutions of (1.1)-(1.4) with $(u_0, v_0) \in \Sigma$, u_0, v_0 bounded in Ω are in $\mathbb{L}^\infty(0, T^*; \mathbb{L}^p(\Omega))$ for all $p \geq 1$.

The proof is an immediate consequence of theorem 3.1, the trivial inequality

$$\int_\Omega (w_1(t,x) + w_2(t,x))^p \, dx \leq L(t) \text{ on } [0, T^*[, \text{ for all } p \geq 1$$

and (2.5).

Under hypothesis of corollary 3.2, if the reactions $f(r,s)$ and $g(r,s)$ are polynomially bounded, then all solutions of (1.1)-(1.3) with $(u_0, v_0) \in \Sigma$, u_0, v_0 bounded in Ω are global in time.

As it has been mentioned above; it suffices to derive an uniform estimate of $\|F_1(w_1, w_2)\|_p$ and $\|F_2(w_1, w_2)\|_p$ on $[0, T^*[$ for some $p > n/2$. Since the functions $f(u,v)$ and $g(u,v)$ are polynomially bounded on Σ, then using relations (2.5) and (2.6) we get that $F_1(w_1, w_2)$ and $F_2(w_1, w_2)$ are too and the proof becomes an immediate consequence of corollary 3.2.

4 Final remarks.

If $\lambda_1\beta_1 \leq a_{11}\beta_1 + a_{12}\beta_2 \leq \lambda_2\beta_1$, then system (1.1)-(1.2) can be written as

$$\frac{\partial v}{\partial t} - a_{22}\Delta v - a_{21}\Delta u = \widetilde{f}(v,u) \text{ in } \mathbb{R}^+ \times \Omega, \qquad (1.1)'$$

$$\frac{\partial u}{\partial t} - a_{12}\Delta v - a_{11}\Delta u = \widetilde{g}(v,u) \text{ in } \mathbb{R}^+ \times \Omega \qquad (1.2)'$$

with the same boundary conditions (1.3) and initial data (1.4) and where

$$\widetilde{f}(v,u) = g(u,v) \text{ and } \widetilde{g}(v,u) = f(u,v) \text{ for all } (u,v) \text{ in } \mathbb{R}^2.$$

In this case, the diffusion-matrix of system becomes

$$A = \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}.$$

Then all previous results remain valid in the region

$$\left\{(v_0, u_0) \in \mathbb{R}^2 \text{ such that } \lambda_1 u_0 \leq a_{12}v_0 + a_{11}u_0 \leq \lambda_2 u_0\right\}$$

which can be written for system (1.1)-(1.2) as

$$\Sigma = \left\{(u_0, v_0) \in \mathbb{R}^2 \text{ such that } \lambda_1 u_0 \leq a_{11}u_0 + a_{12}v_0 \leq \lambda_2 u_0\right\}.$$

$(\widetilde{f}, \widetilde{g})$ points into Σ on $\partial\Sigma$) if

$$\lambda_1 \widetilde{g}(s,r) \leq a_{12}\widetilde{f}(s,r) + a_{11}\widetilde{g}(s,r), \text{ for all } r \text{ and } s \text{ such that } \lambda_1 r = a_{12}s + a_{11}r$$

and

$$a_{12}\widetilde{f}(s,r) + a_{11}\widetilde{g}(s,r) \leq \lambda_2 \widetilde{g}(s,r), \text{ for all } r \text{ and } s \text{ such that } a_{12}s + a_{11}r = \lambda_2 r,$$

which is equivalent to

$$\begin{cases} \lambda_1 f(r,s) \leq a_{11}f(r,s) + a_{12}g(r,s), \text{ for all } r \text{ and } s \text{ such that } \lambda_1 r = a_{11}r + a_{12}s \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{and} \\ a_{11}f(r,s) + a_{12}g(r,s) \leq \lambda_2 f(r,s), \text{ for all } r \text{ and } s \text{ such that } a_{11}r + a_{12}s = \lambda_2 r, \end{cases} \quad (1.6)'$$

and condition (1.8) becomes, for an appropriate constant C_1

$$\widetilde{f}(v,u) + C\widetilde{g}(v,u) \leq C_1 (v + \alpha u + 1) \text{ for all } u \text{ and } v \text{ in } \Sigma,$$

for positive constants C and $\alpha > a_{11} - \lambda_1$ sufficiently close to $a_{11} - \lambda_1$, which can interpreted as

$$Cf(u,v) + g(u,v) \leq C_1 (\alpha u + v + 1) \text{ for all } u \text{ and } v \text{ in } \Sigma, \quad (1.8)'$$

for positive constants C and $\alpha > a_{11} - \lambda_1$sufficiently close to $a_{11} - \lambda_1$.

5 Open Problem

In the case of systems of tripled reaction diffusion equations with a general full matrix of diffusion coefficients and nonhomogeneous boundary conditions, the construction of invariant regions only in terms of the eigenvalues and entries of the diffusion matrix remains an open problem.

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