Int. J. Open Problems Comput. Math. , Vol. 5, No. 1, March, 2012
ISSN 2074-2827; Copyright ©ICSRS Publication, 2012
www.i-csrs.org

Galerkin Method for Solving a Telegraph Equation With a Weighted Integral Condition

A. Guezane-Lakoud [1] and N. Bendjazia [2]

[1,2]Laboratory of Advanced Materials
University Badji Mokhtar B.P. 12, 23000, Annaba. Algeria
e-mail:a_guezane@yahoo.fr

Abstract

The aim of this work is the study of a nonlocal problem for a telegraph equation with weighted integral condition. By the Galerkin method, we construct a discrete numerical solution of the approximate problem, then the convergence of the method and the well posedness of the problem under study are established.

AMS Subject Classifications: 35A35, 35L10, 65M20, 35L20, 35D05, 35N30.

Key words: Galerkin method, A priori estimate, Telegraphe equation, Generalized solution, Integral conditions.

1 Introduction

This paper is devoted to the investigation of a non-local problem for a telegraph equation and a weighted integral condition using the Galerkin method, which is, a convenient tool for both the theoretical and numerical analysis of the considered problem.

More precisely we apply Galerkin method to determine a function $u = u(x,t)$, $(x,t) \in Q = (0,1) \times (0,T)$, which satisfies, in some appropriate sense, the telegraph equation

$$lu = \frac{\partial^2 u}{\partial t^2} - a^2(x,t)\frac{\partial^2 u}{\partial x^2} + c(x,t)u = f(x,t), \tag{1.4}$$

subject to the initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \tag{1.2}$$

$$u(0,t) = 0 \tag{1.3}$$

and the weighted integral condition

$$\int_0^1 g(x)u(x,t)dx = 0 \tag{1.4}$$

where $f \in L_2(Q)$ and φ, $\psi \in L_2(0,1)$ are given functions that satisfy

$$\int_0^1 \varphi(x) = \int_0^1 \psi(x) = 0.$$

We mention that Galerkin method does not give only a discrete approximation scheme, but it provides also a construction proof of the existence of a unique solution. We should mention also here that the presence of the weighted integral term in the boundary condition leads to more difficulties. Integral conditions occur when the values of function on the boundary are related to values inside the domain or when direct measurements on the boundary are not possible.

Problems with integral conditions have many applications in many problems such as the theory of heat conduction, elasticity, heat, plasma physics, control theory, etc... In particular, the presence of integral conditions greatly improves the qualitative and quantitative characteristics of the problem. Many authors studied nonlocal problem with integral conditions by different methods, the reader can see the references therein.

By the help of Galerkin method, Guezane-Lakoud, Dabas and Bahuguna in[11] studied the multidimensional telegraph equation:

$$lu = \frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + bu - c\frac{\partial^2 u}{\partial x^2} = f(x,t), \ (x,t) \in Q = \Omega \times I, \tag{1.5}$$

subject to the initial conditions (1.2) and the integral condition of type

$$\left.\frac{\partial u}{\partial \eta}\right|_{(x,t)\in\partial\Omega\times I} + \int\limits_{\Omega} K(x,\xi)\, u(\xi,t)\, d\xi = 0, \quad \forall x \in \partial\Omega. \tag{1.6}$$

Guezane-Lakoud and Boumaza in [12], investigated the one dimensional case for the equation (1.5) with the the initial conditions (1.2) and the integral condition of type

$$\int_0^1 xu(x,t)dx = 0. \tag{1.7}$$

The first paper investigated nonlocal problem with integral conditions goes back to Cannon [4]. Later, mixed problems with integral conditions were studied by many authors, we can cite the work of Pulkina [14], Beilin[1,2], Bahuguna et al [13], Dabas et al [6].

The summarize of this paper is as follows: In the next section we define the generalized solution and the functional spaces. In section 3 we prove that the generalized solution if it exists is unique. The existence of the generalized solution by using Galerkin method is established in the third section, for this, we construct an approximation solution of the problem (1.1)-(1.4), we prove that we can extract a subsequence which converges to the desired generalized solution.

2 Notation and definition

Let $L^2(0,1)$ be the usual space of Lebesgue square integrable real functions on $(0,1)$ whose inner product and norm will be denoted by $(.,.)$ and $\|.\|$ respectively.

Define the following spaces:
$C_0^2(Q) = \left\{u(x,t) \in C^2(\overline{Q}), \quad u(0,t)=0, \quad \int_0^1 u(x,t)dx = \int_0^1 g(x)u(x,t)dx = 0\right\}$
$C_T^2(Q) = \left\{v(x,t) \in C^2(\overline{Q}), \quad v(x,T)=0, \quad \int_0^1 v(x,t)dx = \int_0^1 g(x)v(x,t)dx = 0\right\}$
The closure of these spaces with respect to the norm

$$\|u\|^2_{H(Q)} = \int_0^T \int_0^1 \left[\left(\int_x^1 u(\xi,t)d\xi \right)^2 + (u(x,t))^2 + \left(\int_x^1 u_t(\xi,t)d\xi \right)^2 \right] dxdt$$

is denoted respectively by $H_0(Q)$ and $H_T(Q)$. Similarly, we obtain the space $H_0(0,1)$ as a closure of the space

$$C_0^2(0,1) = \left\{ w \in C^2[0,1], \quad w(0)=0, \quad \int_0^1 xw(x)dx = 0 \right\}$$

according to the norm

$$\|w\|^2_{H_0(0,1)} = \int_0^1 \left[\left(\int_x^1 w(\xi)d\xi \right)^2 + w^2(x) \right] dx.$$

Let us introduce the notion of a generalized solution. Suppose that u is a solution of problem (1.1)-(1.4), multiply the both sides of equation (1.1) by $\int_x^1 (g(\xi)-g(x))v(\xi,t)\,d\xi$, where $v \in C_T^2(Q)$, integrate by parts the resultant equation over the domain Q, use the initial conditions (1.2) and the fact that $v(x,T)=0$, it yields

$$\int_Q lu \left(\int_x^1 (g(\xi)-g(x))v(\xi,t)\,d\xi \right) dxdt = I_1 - I_2 + I_3$$

where

$$I_1 = \int_Q u_{tt}(x,t)\left(\int_x^1 (g(\xi)-g(x))v(\xi,t)\,d\xi\right)dxdt$$
$$= \int_0^1 g'(x)\left(\int_x^1 \psi(\xi)\,d\xi\right)\left(\int_x^1 v(\xi,0)\,d\xi\right)dx$$
$$+\int_Q g'(x)\int_x^1 u_t d\xi\left(\int_x^1 v_t(\xi,t)\,d\xi\right)dxdt,$$

$$I_2 = \int_Q a^2(x,t)u_{xx}(x,t)\left(\int_x^1 (g(\xi)-g(x))v(\xi,t)\,d\xi\right)dxdt$$
$$= \int_Q a^2 g'(x)uvdxdt$$
$$-\int_Q (4aa_x g'(x)+a^2g^{(2)})u\left(\int_x^1 v(\xi,t)\,d\xi\right)dxdt$$
$$+2\int_Q (aa_x)_x u\left(\int_x^1 (g(\xi)-g(x))v(\xi,t)\,d\xi\right)dxdt$$

and

$$I_3 = \int_Q c(x,t)u(x,t)\left(\int_x^1 (g(\xi)-g(x))v(\xi,t)\,d\xi\right)dxdt =$$
$$-\int_Q cg'(x)\left(\int_x^1 u(\xi,t)\,d\xi\right)\left(\int_x^1 v(\xi,t)\,d\xi\right)dxdt$$
$$+\int_Q c_x\left(\int_x^1 u(\xi,t)\,d\xi\right)\left(\int_x^1 (g(\xi)-g(x))v(\xi,t)\,d\xi\right)dxdt.$$

Summing $I_1 - I_2 + I_3$, we get

$$\int_Q \Big[g'(x)\int_x^1 u_t d\xi\int_x^1 v_t(\xi,t)\,d\xi - a^2g'(x)uv - (4aa_xg'(x) \tag{2.1}$$
$$+a^2g^{(2)})u\int_x^1 v(\xi,t)\,d\xi - cg'(x)\int_x^1 u(\xi,t)\,d\xi\int_x^1 v(\xi,t)\,d\xi$$
$$+\left(c_x - 2\left(aa_x\right)_x\right)\left(\int_x^1 u(\xi,t)\,d\xi\right)\left(\int_x^1 (g(\xi)-g(x))v(\xi,t)\,d\xi\right)\Big]dxdt$$
$$= \int_Q f(x,t)\left(\int_x^1 (g(\xi)-g(x))v(\xi,t)\,d\xi\right)dxdt$$
$$-\int_0^1 g'(x)\left(\int_x^1 \psi(\xi)\,d\xi\right)\left(\int_x^1 v(\xi,0)\,d\xi\right)dx.$$

Definition 2.1 *By a generalized solution of problem (1.1)-(1.4), we mean a function* $u \in H_0(Q)$*satisfying for all* $v \in H_T(Q)$ *identity (2.1) and the condition* $\int_x^1 u(\xi,0)d\xi = \int_x^1 \varphi(\xi)d\xi$.

3 Uniqueness of generalized solution

To study the solvability of problem, we make the following hypotheses

H1: The functions $a(x,t)$ and $c(x,t)$ are nonnegative and satisfy

$$\begin{aligned} 0 &< a_0 \le a(x,t) \le A_0, \quad |a_t, a_x, a_{xx}, a_{xxx}| \le A_1, \\ 0 &< c_0 \le c(x,t) \le C_0, \quad |c_x, c_t| \le C_1. \end{aligned}$$

H2: The function $g(x)$ is nondecreasing and satisfies

$$g \in C^3(0,1), \quad \max_{x\in(0,1)} |g(x)| \le k_1, \quad \max_{x\in(0,1)} \left(|g'(x)|, \left|g^{(2)}(x)\right|, \left|g^{(3)}(x)\right|\right) \le k_2.$$

Theorem 3.1 *Assume that* $f \in L_2(Q)$, φ, $\psi \in L_2(0,1)$ *and hypotheses H1-H2 hold, then the generalized solution of problem (1.1)-(1.4) if it exists is unique.*

Proof. If we suppose that u_1 and u_2 are two different generalized solutions, then the function $u = u_1 - u_2$ is a generalized solution of the problem (1.1)-(1.4) with $f = \varphi = \psi = 0$. We shall prove that $u = 0$ in Q. Let $v \in H_T(Q)$ and denote $Q^\tau = \{(x,t); 0 < x < 1,\ 0 < t \le \tau \le T\}$. Consider the function

$$v(x,t) = \begin{cases} \int_t^\tau u(x,\theta)\, d\theta, & 0 \le t \le \tau \\ 0, & \tau \le t \le T. \end{cases}$$

Substituting v into identity (2.1) and using $v_t(x,t) = -u(x,t)$, it follows that

$$\int_Q \left[g'(x) \int_x^1 u_t d\xi \int_x^1 v_t(\xi,t)\, d\xi - a^2 g'(x) uv \right. \tag{3.1}$$

$$-(4aa_x g'(x) + a^2 g^{(2)})u \int_x^1 v(\xi,t)\, d\xi - cg'(x) \int_x^1 u(\xi,t)\, d\xi \int_x^1 v(\xi,t)\, d\xi$$

$$\left. + (c_x - 2(aa_x)_x) \left(\int_x^1 u(\xi,t)\, d\xi \right) \left(\int_x^1 (g(\xi) - g(x)) v(\xi,t)\, d\xi \right) \right] dxdt = 0$$

Integrating by parts yields

$$\int_0^1 \left[g'(x) \left(\int_x^1 u(\xi,\tau)\, d\xi \right)^2 + g'(x) a^2(x,0)(v(x,0))^2 \right. \tag{3.2}$$

$$+g'(x)c\left(x,0\right)\left(\int_x^1 v\left(\xi,0\right)d\xi\right)^2\Bigg]dx$$

$$=\int_{Q^\tau}\left[2aa_t g'(x)\left(v\right)^2+2[6aa_x g^{(2)}(x)+a^2g^{(3)}-6\left(\left(a_x^2\right)+aa_{xx}\right)g'\right]$$

$$\times\int_x^1 ud\xi\int_x^1 v\left(\xi,t\right)d\xi+2(4aa_xg'-a^2g^{(2)})v\int_x^1 ud\xi$$

$$-2\left(c_x+2\left(3aa_{xx}+aa_{xxx}\right)\right)\int_x^1 u\left(x,t\right)d\xi\int_x^1\left(g(\xi)-g(x)\right)v\left(x,t\right)d\xi$$

$$-g'(x)c_t\left(\int_x^1 vd\xi\right)^2\Bigg]dxdt.$$

Applying Cauchy inequality to the the right hand side of (3.2), using condition H1 and H2 and $\epsilon-$inequality for $\varepsilon=1$, we get

$$\int_0^1 k_2\left[\left(\int_x^1 u\left(\xi,\tau\right)d\xi\right)^2dx+a^2\left(x,0\right)\left(v\left(x,0\right)\right)^2+c\left(x,0\right)\left(\int_x^1 v\left(\xi,0\right)d\xi\right)^2\right]dx$$

$$\leq\int_{Q^\tau}\left[k_2(6A_0A_1+A_0^2)\left|v\right|^2+[\left(6A_1^2+12A_0A_1+A_0^2+C_1\right)k_2\right. \tag{3.3}$$

$$+(8A_0A_1+C_1)k_1]\left(\int_x^1 v\left(\xi,t\right)d\xi\right)^2$$

$$\left.+[\left(6A_1^2+24A_0A_1+2A_0^2\right)k_2+(8A_0A_1+C_1)k_1]\left(\int_x^1 ud\xi\right)^2\right]dxdt.$$

Thus

$$\int_0^1\left[\left(\int_x^1 u\left(\xi,\tau\right)d\xi\right)^2dx+\left(v\left(x,0\right)\right)^2+\left(\int_x^1 v\left(\xi,0\right)d\xi\right)^2\right]dx$$

$$\leq L\int_{Q^\tau}\left[\left|v(x,t)\right|^2+\left(\int_x^1 v\left(\xi,t\right)d\xi\right)^2+\left(\int_x^1 ud\xi\right)^2\right]dxdt. \tag{3.4}$$

where

$$L=\max\left\{(6A_0A_1+A_0^2)k_2,\left(6A_1^2+12A_0A_1+A_0^2+C_1\right)k_2+(8A_0A_1+C_1)k_1,\right.$$
$$\left.\left(6A_1^2+24A_0A_1+2A_0^2\right)k_2+(8A_0A_1+C_1)k_1\right\}/\min\left\{k_2,k_2a_0^2,k_2c_0\right\}.$$

Let us introduce the function $w(x,t)=\int_0^1 u(x,\theta)\,d\theta$, we see that

$$v(x,t)=w(x,\tau)-w(x,t),\quad v(x,0)=w(x,\tau),$$
$$v^2(x,t)\le 2w^2(x,\tau)+2w^2(x,t),$$

consequently inequality (3.4) becomes

$$\int_0^1\left[\left(\int_x^1 u(\xi,\tau)\,d\xi\right)^2+(1-2K\tau)\left(w^2(x,\tau)+\left(\int_x^1 w(\xi,\tau)\,d\xi\right)^2\right)\right]dx$$

$$\le 2L\int_{Q^\tau}\left[\left(\int_x^1 u(\xi,t)\,d\xi\right)^2 d\xi+w^2(x,t)+\left(\int_x^1 w(\xi,t)\right)^2\right]dxdt. \tag{3.5}$$

Since τ is arbitrary, let $0<\tau<\dfrac{1}{2L}$, then (3.5) becomes

$$\int_0^1\left[\left(\int_x^1 u(\xi,\tau)\,d\xi\right)^2+\left(w^2(x,\tau)+\left(\int_x^1 w(\xi,\tau)\,d\xi\right)^2\right)\right]dx$$
$$\le L'\int_{Q^\tau}\left[\left(\int_x^1 u(\xi,t)\,d\xi\right)^2 d\xi+w^2(x,t)+\left(\int_x^1 w(\xi,t)\right)^2\right]dxdt$$

where $L'=\frac{2L}{(1-2L\tau)}$. Applying Gronwall Lemma, it yields

$$\int_0^1\left[\left(\int_x^1 u(\xi,\tau)\,d\xi\right)^2+w^2(x,\tau)+\left(\int_x^1 w(\xi,\tau)\,d\xi\right)^2\right]dx\le 0,$$

hence $u(x,\tau)=0$, for all $x\in(0,1)$ and $\tau\in\left(0,\frac{1}{2L}\right)$. If $T\le\frac{1}{2L}$, then $u=0$ in Q. If $T\ge\frac{1}{2L}$ we get $]0,T[\subset\cup_{n=1}^{n=n_0}\left(\frac{n-1}{2L},\dfrac{n}{2L}\right)$, where $n_0=[2LT]+1$, $[2LT]$ is the entire part of $2LT$, then repeating the same argument for $\tau\in\left(\frac{n-1}{2L},\frac{n}{2L}\right)$, we get $u(x,\tau)=0$ in Q. Thus, the uniqueness is proved.

4 Existence of generalized solution

In order to prove the existence of the generalized solution we apply Galerkin method.

Theorem 4.1 *Assume that the assumptions of Theorem 3.1 hold, then the non-local problem (1.1)-(1.4) has a unique solution* $u\in H_0(Q)$

Proof. Let $\{w_k(x)\}$ be a fundamental system in $H_0(0,1)$, such that $(w_k, w_i) = \delta_{k,i}$, then the approximate solution of the problem (1.1)-(1.4) can be written as

$$u^{(n)} = \sum_{k=1}^{n} \alpha_k(t)\, w_k(x). \tag{4.1}$$

The approximate of the functions $\varphi(x)$ and $\psi(x)$ are denoted respectively by

$$\varphi^{(n)}(x) = \sum_{k=1}^{n} \varphi_k w_k(x), \quad \psi^{(n)}(x) = \sum_{k=1}^{n} \psi_k w_k(x)\, \alpha_k(0) = \varphi_k, \quad \alpha_k'(0) = \psi_k. \tag{4.2}$$

Substituting the approximate solution in equation (1.1), multiplying both sides by $\int_x^1 (g(\xi) - g(x))w_i(\xi)d\xi$, then integrating according to x on $(0,1)$, it yields

$$\int_0^1 (u_{tt}^{(n)}(x,t) - a^2(x,t)u_{xx}^{(n)}(x,t) + c(x,t)u^{(n)}(x,t))\times$$

$$\left(\int_x^1 (g(\xi) - g(x))w_i(\xi)d\xi\right) dx = \int_0^1 f(x,t)\left(\int_x^1 (g(\xi) - g(x))w_i(\xi)d\xi\right) dx. \tag{4.3}$$

In view of (4.1), we get

$$\begin{aligned}
&\sum_{k=1}^{n} \alpha_k''(t)\left(w_k(x), \int_x^1 (g(\xi) - g(x))w_i(\xi)\,d\xi\right)_{L_2(0,1)} + \\
&+\sum_{k=1}^{n} \alpha_k(t)\Bigg[\left(-a^2(x,t)w_k''(x), \int_x^1 (g(\xi) - g(x))w_i(\xi)d\xi\right)_{L_2(0,1)} \\
&\qquad +\left(c(x,t)w_k(x), \int_x^1 (g(\xi) - g(x))w_i(\xi)d\xi\right)_{L_2(0,1)}\Bigg] \\
&= \int_0^1 f(x,t)\int_x^1 (g(\xi) - g(x))w_i(\xi)d\xi dx.
\end{aligned} \tag{4.4}$$

Integrating by parts in $L_2(0,1)$ the left-hand side of (4.4), we obtain

$$\begin{aligned}
&\sum_{k=1}^{n} \alpha_k''(t) + \left[g'(x)\left(\int_x^1 w_k(\xi)\,d\xi, \int_x^1 w_i(\xi)\,d\xi\right)_{L_2(0,1)}\right] \\
&\qquad +\sum_{k=1}^{n} \alpha_k(t)\,[c_x - 2(aa_{xxx} + 3aa_{xx})\times \\
&\left(\int_x^1 w_k(\xi)\,d\xi, \int_x^1 (g(\xi) - g(x))w_i(\xi)d\xi\right)_{L_2(0,1)}
\end{aligned}$$

$$+(6aa_x g^{(2)}(x) + a^2 g^{(3)}(x) - 6\left(\left(a_x^2\right) + aa_{xx}\right) g'(x))$$
$$\left(\int_x^1 w_k\left(\xi\right) d\xi, \int_x^1 w_i(\xi) d\xi\right)_{L_2(0,1)} \times$$
$$\left.(4aa_x g' + a^2 g^{(2)}) \left(w_k, \int_x^1 w_i(\xi) d\xi\right)_{L_2(0,1)} + g' a^2 \left(w_k, w_i\right)_{L_2(0,1)}\right]$$
$$= \int_0^1 f(x,t) \int_x^1 (g(\xi) - g(x)) w_i(\xi) d\xi dx. \tag{4.5}$$

Let us make the following notation:

$$\theta_{k,i} = g'(x)(\int_x^1 w_k\left(\xi\right) d\xi, \int_x^1 w_i(\xi) d\xi)_{L_2(0,1)}$$

$$\sigma_{k,i} = \left(c_x + 2\left(3aa_{xx} + aa_{xxx}\right)\right) \times$$
$$\left(\int_x^1 w_k\left(\xi\right) d\xi, \int_x^1 (g(\xi) - g(x)) w_i(\xi) d\xi\right)_{L_2(0,1)}$$
$$(6aa_x g^{(2)}(x) + a^2 g^{(3)}(x) - 6\left(\left(a_x^2\right) + aa_{xx}\right) g'(x))$$
$$\times \left(\int_x^1 w_k\left(\xi\right) d\xi, \int_x^1 w_i(\xi) d\xi\right)_{L_2(0,1)}$$
$$\left.+ \, (4aa_x g' + a^2 g^{(2)}) \left(w_k, \int_x^1 w_i(\xi) d\xi\right)_{L_2(0,1)} + g' a^2 \left(w_k, w_i\right)_{L_2(0,1)}\right]$$

$$f_i(t) = \int_0^1 f(x,t) \int_x^1 (g(\xi) - g(x)) w_i(\xi) d\xi dx,$$

then (4.5) becomes

$$\begin{aligned} \sum_{k=1}^{n} \alpha_k''\left(t\right) \theta_{k,i} + \alpha_k\left(t\right) \sigma_{k,i} &= f_i(t) \\ \alpha_k\left(0\right) &= \varphi_k; \quad \alpha_k'\left(0\right) = \psi_k \end{aligned}$$

Consequently we obtain a Cauchy system of second order linear differential equations with smooth coefficients, so it has one and only one solution, that for every n there exists a unique sequence $u^{(n)}$ that satisfies (4.3).

Now we shall prove that this sequence is convergent, for this, we will prove that is bounded and so we can extract a subsequence that is weakly convergent, then its limit is the desired solution of the problem (1.1)-(1.4).

Lemma 4.2 *The sequence* $\left(u^{(n)}\right)$ *is bounded.*

Proof. Multiplying (4.3) by $\alpha_i^{'}(t)$ then summing with respect to i from 1 to n it yields

$$\int_0^1 \left(u_{tt}^{(n)}(x,t) - a^2(x,t)u_{xx}^{(n)}(x,t) + c(x,t)u^{(n)}(x,t)\right) \times \int_x^1 (g(\xi)-g(x))u_t^{(n)}(\xi,t)d\xi dx$$
$$= \int_0^1 f(x,t) \int_x^1 (g(\xi)-g(x))u_t^{(n)}(\xi,t)d\xi dx \tag{4.6}$$

Integrating (4.6) over t from 0 to τ and applying similar technics that have been used to prove the uniqueness, we obtain

$$\int_0^1 \left[\frac{1}{2}g'(x)c\left(x,\tau\right)\left(\int_x^1 u^{(n)}(\xi,\tau)d\xi\right)^2 + \frac{1}{2}g'(x)a^2\left(x,\tau\right)\left(u^{(n)}(x,\tau)\right)^2 + \frac{1}{2}g'(x)\left(\int_x^1 u_t^{(n)}(\xi,\tau)d\xi\right)^2\right]dx$$
$$= \int_0^1 \left[\frac{1}{2}g'(x)c\left(x,0\right)\left(\int_x^1 \varphi^{(n)}(\xi)d\xi\right)^2 + \frac{1}{2}g'(x)a^2\left(x,0\right)\left(\varphi^{(n)}(x)\right)^2 + \frac{1}{2}g'(x)\left(\int_x^1 \psi^{(n)}(\xi)d\xi\right)^2 dx\right] +$$
$$\int_{Q^\tau} -2(a_x^2 + aa_{xx})u^{(n)}\int_x^1 (g(\xi)-g(x))u_t^{(n)}(\xi,t)d\xi + a^2g^{(2)}u^{(n)}\int_x^1 u_t^{(n)}(\xi,t)d\xi$$
$$+g'(x)aa_t(u^{(n)})^2 + c_x\left(\int_x^1 u^{(n)}(\xi,t)d\xi\right)\left(\int_x^1 (g(\xi)-g(x))u_t^{(n)}(\xi,t)d\xi\right)$$
$$+\frac{1}{2}g'(x)c_t\left(\int_x^1 u^{(n)}(\xi,t)d\xi\right)^2 + f(x,t)\left(\int_x^1 (g(\xi)-g(x))u_t^{(n)}(\xi,t)d\xi\right)\Bigg]dxdt.$$

With the help of Cauchy inequality, we obtain

$$\int_0^1 k_2\left[\left(\int_x^1 u^{(n)}(\xi,\tau)d\xi\right)^2 + u^{(n)}(x,\tau)^2 + \left(\int_x^1 u_t^{(n)}(\xi,\tau)d\xi\right)^2\right]dx$$
$$\leq \widetilde{L}\left[\left\|\varphi^{(n)}\right\|^2 + \left\|\psi^{(n)}\right\|^2 + \parallel f\parallel_{L_2(Q^\tau)}^2 + \int_{Q^\tau} C_1(k_1+1/2k_2)\left(\int_x^1 u^{(n)}(\xi,t)d\xi\right)^2\right.$$
$$+[k_1(C_1+2A_1^2+2A_0A_1+1)+k_2A_0^2]\left(\int_x^1 u_t^{(n)}(\xi,t)d\xi\right)^2$$
$$\left.[2k_1(A_0A_1+A_1^2)+k_2(A_0^2+A_0A_1)]u^{(n)}(x,t)^2dxdt\right],$$

where $\widetilde{L} = \widetilde{M} \ / \widetilde{m}, \ \ \widetilde{m} \ = \min \{k_2, k_2 a_0^2, k_2 c_0\}$

$\widetilde{M} = \max \{k_2, k_2 A_0^2, k_2 C_0, C_1(k_1 + 1/2k_2), k_1(C_1 + 2A_1^2 + 2A_0 A_1 + 1) +$

$k_2 A_0^2, 2k_1(A_0 A_1 + A_1^2) + k_2(A_0^2 + A_0 A_1).\}$

Now applying Gronwall Lemma, we get

$$\int_0^1 \left[\left(\int_x^1 u^{(n)}(\xi, \tau) d\xi \right)^2 + u^{(n)}(x, \tau)^2 + \left(\int_x^1 u_t^{(n)}(\xi, \tau) d\xi \right)^2 \right] dx$$
$$\leq e^{\tau \widetilde{L}} \left(\left\| \varphi^{(n)} \right\|^2 + \left\| \psi^{(n)} \right\|^2 + \| \ f \ \|_{L_2(Q)}^2 \right) \tag{4.7}$$

Integrating (4.7) according to τ on $[0, T]$ it yields

$$\|u\|_{H(Q)}^2 \leq e^{T \widetilde{L}} \left(\left\| \varphi^{(n)} \right\|^2 + \left\| \psi^{(n)} \right\|^2 + \| \ f \ \|_{L_2(Q)}^2 \right). \tag{4.8}$$

The inequality (4.8) implies the boundeness of the sequence $u^{(n)}$.

4.3 *We have proved that the sequence* $\left\{ u^{(n)} \right\}$ *is bounded, so we can extract a subsequence which we denote by* $\left\{ u^{(n_k)} \right\}$, *that is weakly convergent, then we prove that its limit is the desired solution of the problem (1.1)-(1.4).*

Lemma 4.4 *The limit of the subsequence* $\left\{ u^{(n_k)} \right\}$ *is the solution of the problem (1.1)-(1.4).*

Proof. We shall prove that the limit of the subsequence $\left\{ u^{(n_k)} \right\}$ satisfies the identity (2.1) for all function $v = \sum_{i=1}^n \theta_i(t) w_i(x) \in H_T(Q)$. Since the set $S_n = \{v(x,t) = \sum_{k=1}^n v_k(t) w_k(x), \ v_k(t) \in C^2(0,T), v_k(T) = 0\}$ is such that $\cup_{n=1}^\infty S_n$ is dense in $H_T(Q)$, it suffice to prove (2.1) for $v \in S_n$.

Multiplying (4.3) by $v_k(t)$, summing according to k from 1 to n then integrating over t from 0 to T, we obtain

$$\int_Q \left[g'(x) \int_x^1 u_t^{(n_k)} d\xi \int_x^1 v_t(\xi, t) \, d\xi - a^2 g'(x) u^{(n_k)} v \right. \tag{4.9}$$

$$-(4aa_x g'(x) + a^2 g^{(2)}) u^{(n_k)} \int_x^1 v(\xi, t) \, d\xi - c g'(x) \int_x^1 u^{(n_k)}(\xi, t) \, d\xi \int_x^1 v(\xi, t) \, d\xi$$
$$\left. + \left(c_x - 2 \left(a a_x \right)_x \right) \left(\int_x^1 u^{(n_k)}(\xi, t) \, d\xi \right) \left(\int_x^1 (g(\xi) - g(x)) v(\xi, t) \, d\xi \right) \right] dx dt$$
$$= \int_Q f(x,t) \left(\int_x^1 (g(\xi) - g(x)) v(\xi, t) \, d\xi \right) dx dt$$
$$- \int_0^1 g'(x) \left(\int_x^1 \psi^{(n_k)}(\xi) \, d\xi \right) \left(\int_x^1 v(\xi, 0) \, d\xi \right) dx.$$

Let u be the weak limit of the subsequence $\left\{ u^{(n_k)} \right\}$. Letting k to infinity, we get that the limit function u satisfies (2.1) for every $v^n \in H_T(Q)$, since $\overline{\cup_{n=1}^\infty S_n} = H_T(Q)$ and so u is a generalized solution of the problem (1.1)-(1.4).

5 Open Problem

In this paper we have studied a one-dimensional telegraph equation with a weighted integral condition. One can apply numerical methods like the homotopy analysis method (HAM), variational iteration method, ..., to the same problem.

It will be interesting if the Galerkin method is applied to the same equation but with more general integral conditions.

References

[1] S. A. Beilin, On a Mixed nonlocal problem for a wave equation, Electron. J. Differential Equations 103, (2006), 1–10.

[2] S. A. Belin, Existence of solutions for one dimensional wave equations with nonlocal conditions, Electronic Journal of Differential equations. 76 (2001), 1–8.

[3] D. Bahuguna, S. Abbas and J. Dabas, Partial functional differential equation with an integral condition and applications to population dynamics, Nonlinear Anal, TMA 69 (2008), 2623–2635.

[4] J.R. Cannon, The solution of heat equation subject to the specification of energy. Quart. Appl. Math. 1963, Vol 21, N 2, 155–160.

[5] J.R. Cannon and Y. Lin, A Galerkin procedure for diffusion equations subject to specification of mass, SIAM J. Numer. Anal. 24 (1987) 499–515.

[6] J. Dabas and D. Bahuguna, An integro-differential equation with an integral boundary condition, Mathematical and Computer Modelling 50 (2009), 123–131.

[7] D. Dao-Quing and H. Yu, Remarks on heat equation with integral boundary conditions, Nonlinear Analysis, TMA. 67 (2) (2007) 468–475.

[8] G. Fairweathe and J. C. Lopez-Marcos, Galerkin methods for a semilinear parabolic problem with nonlocal boundary conditions, Adv. Comput. Math., 6, 243–262 (1996).

[9] A. Guezane-Lakoud, A. Chaoui, Rothe method applied to semilinear hyperbolic integro-differential equation with integral conditions, IJOPCM, Vol. 4, No. 1, 1–14, (2011).

[10] A. Guezane-Lakoud, D. Belakroum, Rothe's method for a telegraph equation with integral conditions, Nonlinear Anal. 70 (2009) 3842–3853.

[11] A. Guezane-Lakoud, J. Dabas, and D. Bahuguna, Existence and uniqueness of generalized solutions to a telegraph equation with an integral boundary condition via Galerkin method, Int. J. Math. Math. Sc, Vol 2011 (2011), Art ID 451492, 14pp.

[12] A. Guezane-Lakoud, N. Boumaza, Galerkin method applied to non local problem; Int. J. Appl. Math. Stat, 19, (2010), 72–89.

[13] N.I. Ionkin, Solutions of boundary value problem in heat conduction theory with nonlocal boundary conditions, Differents.Urav. 13 (1977), 294–304.

[14] L. S. Pulkina, Non local problem with a first kind integral condition for a multidimensional hyperbolic equation, Doklady Akademii, (2007), Vol 416, N 5, 597-599.

[15] H.Yao and Y. Lin, New algorithm for solving a nonlinear hyperbolic telegraph equation with an integral condition, Int. J. Numer. Meth. Biomed. Engng. (2010), DOI: 10.1002/cnm.1376

[16] H. Yao, Reproducing Kernel Method for the Solution of Nonlinear Hyperbolic Telegraph Equation with an Integral Condition, Numerical Methods for Partial Differential Equations, Volume 27, Issue 4, pages 867–886, July 2011.