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Modified Noor iterative procedure for three asymptotically pseudocontractive maps

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Abstract

Rafiq, Acu and Sofonea [9] recently considered an iterative algorithm for two asymptotically pseudocontractive mappings. They proposed that it is an open problem to show that their results can be extended to the case of three mappings. This paper answers the question in the affirmative.

Keywords: *Modified Noor iteration; asymptotically pseudocontractive mappings; Banach spaces.*

1 Introduction

Let E be an arbitrary real Banach Space and let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{f \in E^* :< x, f >= \|x\|^2 = \|f\|^2\}, \forall x \in E \qquad (*)$$

where E^* denotes the dual space of E and $< .,. >$ denotes the generalized duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j.
Remark 1.1 The above J satisfies

$$< x, j(y) >\leq \|x\|\|y\|, \forall x \in E, \forall j(y) \in J(y)$$

.
Proof. Denote $j(y)$ by f. Since $f \in E^*$, we have

$$< x, j(y) >\leq \|x\|\|y\|.$$

From (*), we know that $\|f\| = \|y\|$. Hence the result holds.
In 1972, Goebel and Kirk [4] introduced the class of asymptotically nonexpansive mappings as follows:
Let K be a nonempty closed convex subset of E and $T : K \to K$ be a map.
Definition 1.2 A mapping T is said to be asymptotically nonexpansive if for each $x, y \in K$

$$< T^n x - T^n y, j(x-y) > \leq k_n \|x-y\|^2, \forall n \geq 0,$$

where sequence $(k_n) \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$.
Definition 1.3 The mapping T is said to be uniformly L- Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x-y\|,$$

for any $x, y \in K$ and $\forall n \geq 0$.
Definition 1.4 The mapping T is said to be asymptotically pseudocontractive if there exists a sequence $(k_n) \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and for any $x, y \in K$ there exists $j(x-y) \in J(x-y)$ such that

$$< T^n x - T^n y, j(x-y) > \leq k_n \|x-y\|^2, \forall n \geq 0.$$

Remark 1.5 One can see that, every operator which is asymptotically pseudocontractive, in general, may not admit fixed point. The existence of fixed point result for asymptotically pseudocontractive maps depends on the space, nature of subset and further properties of the operator. See, [1- 4].
The concept of asymptotically pseudocontractive mappings was introduced by Schu[10].
Chang et al. [3] proved a strong convergence theorem for a pair of L-Lipschitzian mappings instead of a single map used in [8]. In fact, they proved the following theorem :
Theorem 1.6 ([3]). Let E be a real Banach space, K be a nonempty closed convex subset of E, $T_i : K \to K$, $(i = 1, 2)$ be two uniformly L_i-Lipschitzian mappings with $F(T_1) \cap F(T_2) \neq \phi$, where $F(T_i)$ is the set of fixed points of T_i in K and ρ be a point in $F(T_1) \cap F(T_2)$. Let $k_n \subset [1, \infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences in $[0, 1]$ satisfying the following conditions:
(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ (iii) $\sum_{n=1}^{\infty} \beta_n < \infty$
(iv) $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.
For any $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_n$$
$$y_n = (1-\beta_n)x_n + \beta_n T_2^n x_n.$$

If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T_1^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|)$$

for all $j(x - \rho) \in J(x - \rho)$ and $x \in K$, (i = 1, 2), then $\{x_n\}_{n=1}^{\infty}$ converges strongly to ρ.
The result above extends and improves the corresponding results of [8] from one uniformly Lipschitzian asymptotically pseudocontractive mapping to two uniformly Lipschitzian asymptotically pseudocontractive mappings. In fact, if the iteration parameter $\{\beta_n\}_{n=0}^{\infty}$ in Theorem 1.1 above is equal to zero for all n and $T_1 = T_2 = T$ then, we have the main result of Ofoedu [8].
Recently, Rafiq, Acu and Sofonea [9], improved the results of Chang et al. [3] in a significant more general context. They then gave an open problem whether their results can be extended for the case of three mappings which are more general than the two maps. Indeed, they proved the following theorem.

Theorem 1.7 ([9]). Let K be a nonempty closed convex subset of a real Banach space E, $T_i : K \to K$, $(i = 1, 2)$ be two uniformly L-Lipschitzian mappings with sequence $k_n \subset [1, \infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ such that $F(T_1) \cap F(T_2) \neq \phi$, where $F(T_i)$ is the set of fixed points of T_i in K and ρ be a point in $F(T_1) \cap F(T_2)$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, $lim_{n\to\infty}\alpha_n = \beta_n = 0$. For any $x_0 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence iteratively defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_1^n y_n \\ y_n &= (1 - \beta_n)x_n + \beta_n T_2^n x_n. \end{aligned} \tag{1.1}$$

Suppose there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T_i^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|), \forall x \in K (i = 1, 2)$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\rho \in F(T_1) \cap F(T_2)$.

The purpose of this paper is to address the open problem of [9] by following their style and technique of proof. For this, we need the following concepts and Lemmas.
Definition 1.8 (see, [6, 7]). Let $T_1, T_2, T_3 : K \to K$ be three mappings. For any given $x_1 \in K$, the modified Noor iteration $\{x_n\}_{n=1}^{\infty} \subset K$ is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1^n y_n$$

$$y_n = (1 - \beta_n)x_n + \beta_n T_2^n z_n$$

$$z_n = (1-\gamma_n)x_n + \gamma_n T_3^n x_n, \quad n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are three real sequences satisfying some conditions. If $\gamma_n = 0$, we define modified Ishikawa iteration procedure $\{x_n\}_{n=1}^{\infty}$ by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1^n y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T_2^n x_n, \quad n \geq 0. \end{aligned} \tag{1.3}$$

Also, if $\beta_n = 0$, we define modified Mann iteration procedure $\{x_n\}_{n=1}^{\infty}$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n x_n, \quad n \geq 1. \tag{1.4}$$

Lemma 1.9 [1, 9]. Let E be real Banach Space and $J : E \to 2^{E^*}$ be the normalized duality mapping.Then,for any $x, y \in E$

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, j(x+y) >, \forall j(x+y) \in J(x+y).$$

Lemma 1.10 [5]. Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying

$$\sum_{n=0}^{\infty} b_n = +\infty \text{ and } \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n \Phi(a_{n+1}), \quad \forall n \geq N_0$$

where $\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, then $\lim_{n\to\infty} a_n = 0$.

2 Main results

Theorem 2.1. Let E be a real Banach space, K be a nonempty closed convex subset of E and $T_i : K \to K$, $(i = 1, 2, 3)$ be three uniformly L_i-Lipschitzian mappings with $\rho \in F(T_1) \cap F(T_2) \cap F(T_3) \neq \phi$. Let $k_n \subset [1,\infty)$ be a sequence with $k_n \to 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be three sequences in $[0, 1]$ satisfying the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$, $lim_{n\to\infty}\alpha_n = \beta_n = \gamma_n = 0$. For any $x_1 \in K$, define the sequence $\{x_n\}_{n=1}^{\infty}$ by the iterative procedure (1.2). Suppose there exists a strictly increasing function

$\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T_i^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|), \forall x \in K (i = 1, 2, 3). \quad (2.1)$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\rho \in F(T_1) \cap F(T_2) \cap F(T_3)$.

Proof. Since T_1, T_2 and T_3 are uniformly L_i-Lipschitzian mappings, we have that for all $x, y \in K$

$$\|T_i^n x - T_i^n y\| \leq L_i \|x - y\|, (i = 1, 2, 3).$$

For convenience, denote $L=\max\{L_1, L_2, L_3\}$.

By $lim_{n\to\infty}\alpha_n = \beta_n = \gamma_n = 0$ and $lim_{n\to\infty} k_n = 1$, there exists $n_0 \in N$ such that $\forall n \geq n_0$,

$$\begin{array}{lll} \alpha_n & \leq & min\{\frac{1}{2+3L}, \frac{\Phi(2(\Phi^{-1}(a_0)))}{18(1+L)(2+3L)(\Phi^{-1}(a_0))^2}\}, \\ \beta_n & \leq & min\{\frac{1}{1+2L}, \frac{\Phi(2(\Phi^{-1}(a_0)))}{18L(2+3L)\Phi(2(\Phi^{-1}(a_0))}\}, \\ \gamma_n & \leq & \frac{1}{2}\{\frac{1}{1+L}\} \end{array} \quad (2.2)$$

and

$$k_n - 1 \quad \leq \quad \frac{\Phi(2(\Phi^{-1}(a_0))}{54(\Phi^{-1}(a_0))^2}. \quad (2.3)$$

Define $a_{0,i} = \|x_{n_0} - T_1^{n_0} x_{n_0}\| \|x_{n_0} - \rho\| + (k_n - 1)\|x_{n_0} - \rho\|^2, (i = 1, 2, 3)$ such that $a_0 = max\{a_{0,1}, a_{0,2}, a_{0,3}\}$. Thus, by (2.1)

$$< T_i^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|).$$

So that, on simplifying

$$\|x_n - \rho\| \leq \Phi^{-1}(a_0). \quad (2.4)$$

Now, we claim that $\|x_n - \rho\| \leq 2\Phi^{-1}(a_0), \forall n \geq n_0$. Clearly inview of (2.4), the claim holds for $n = 0$. We next assume that $\|x_n - \rho\| \leq 2\Phi^{-1}(a_0)$, for some n and we shall prove that $\|x_{n+1} - \rho\| \leq 2\Phi^{-1}(a_0)$. Suppose this is not true, i.e. $\|x_{n+1} - \rho\| > 2\Phi^{-1}(a_0)$.

Thus, we have from (1.2)

$$\begin{array}{lll} \|z_n - \rho\| & = & \|(1 - \gamma_n)x_n + \gamma_n T_3^n x_n - \rho\| \\ & \leq & \|x_n - \rho\| + \gamma_n \|T_3^n x_n - x_n\| \\ & \leq & \|x_n - \rho\| + \gamma_n (1 + L)\|x_n - \rho\| \\ & \leq & 3\Phi^{-1}(a_0). \end{array} \quad (2.5)$$

Similarly,

$$\begin{aligned} \|y_n-\rho\| &= \|(1-\beta_n)x_n+\beta_n T_2^n z_n-\rho\| \\ &\le \|x_n-\rho\|+\beta_n\|T_2^n z_n-x_n\| \\ &\le \|x_n-\rho\|+\beta_n(L\|z_n-\rho\|+\|x_n-\rho\|) \\ &\le 2\Phi^{-1}(a_0)+(2+3L)\beta_n\Phi^{-1}(a_0) \\ &\le 3\Phi^{-1}(a_0). \end{aligned} \tag{2.6}$$

Also, we have the following estimates:

$$\begin{aligned} (i)\ \ \|T_1^n y_n-x_n\| &\le \|x_n-\rho\|+\|T_1^n y_n-\rho\| \\ &\le \|x_n-\rho\|+L\|y_n-\rho\| \\ &\le (2+3L)\Phi^{-1}(a_0), \\ (ii)\ \ \|x_{n+1}-\rho\| &\le 3\Phi^{-1}(a_0), \\ (iii)\ \ \|x_{n+1}-x_n\| &\le (2+3L)\alpha_n\Phi^{-1}(a_0), \\ (iv)\ \ \|x_n-T_2^n z_n\| &\le (2+3L)\Phi^{-1}(a_0), \end{aligned}$$

$$\begin{aligned} (v)\ \ \|y_n-x_{n+1}\| &\le \|y_n-x_n\|+\|x_{n+1}-x_n\| \\ &\le \beta_n\|T_2^n z_n-x_n\|+\|x_{n+1}-x_n\| \\ &\le (2+3L)\beta_n\Phi^{-1}(a_0)+(2+3L)\alpha_n\Phi^{-1}(a_0). \end{aligned} \tag{2.7}$$

Using Lemma 1.9 and the above estimates, we have

$$\begin{aligned} \|x_{n+1}-\rho\|^2 &= \|(1-\alpha_n)x_n+\alpha_n T_1^n y_n-\rho\|^2 \\ &= \|x_n-\rho+\alpha_n(T_1^n y_n-x_n)\|^2 \end{aligned}$$

$$\begin{aligned} &\le \|x_n-\rho\|^2-2<x_n-T_1^n y_n, j(x_{n+1}-\rho)> \\ &= \|x_n-\rho\|^2+2\alpha_n<T_1^n x_{n+1}-\rho, j(x_{n+1}-\rho)> \\ &\quad -2\alpha_n<x_{n+1}-\rho, j(x_{n+1}-\rho)> \\ &\quad +2\alpha_n<T_1^n y_n-T_1^n x_{n+1}, j(x_{n+1}-\rho)> \\ &\quad +2\alpha_n<x_{n+1}-x_n, j(x_{n+1}-\rho)> \\ &\le \|x_n-\rho\|^2+2\alpha_n(k_n\|x_n-\rho\|^2-\Phi(\|x_n-\rho\|)) \\ &\quad -2\alpha_n\|x_{n+1}-\rho\|^2+2\alpha_n\|T_1^n y_n-T_1^n x_{n+1}\|\|x_{n+1}-\rho\| \\ &\quad +2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\ &= \|x_n-\rho\|^2+2\alpha_n(k_n-1)\|x_n-\rho\|^2-2\alpha_n\Phi(\|x_n-\rho\|)) \\ &\quad +2\alpha_n L\|y_n-x_{n+1}\|\|x_{n+1}-\rho\| \\ &\quad +2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\ &\le \|x_n-\rho\|^2-2\alpha_n\Phi(2(\Phi^{-1}(a_0))+2\alpha_n(k_n-1)\|x_n-\rho\|^2 \\ &\quad +2\alpha_n L((2+3L)\beta_n\Phi^{-1}(a_0)+(2+3L)\alpha_n\Phi^{-1}(a_0))\|x_{n+1}-\rho\| \\ &\quad +2\alpha_n((2+3L)\alpha_n\Phi^{-1}(a_0))\|x_{n+1}-\rho\| \\ &\le \|x_n-\rho\|^2-2\alpha_n\Phi(2(\Phi^{-1}(a_0))+18\alpha_n(k_n-1)(\Phi^{-1}(a_0))^2 \\ &\quad +6\alpha_n\beta_n L((2+3L)(\Phi^{-1}(a_0))^2+6\alpha_n^2(1+L))(2+3L)(\Phi^{-1}(a_0))^2 \end{aligned} \tag{2.8}$$

Inview of (2.2) and (2.3), equation (2.8) becomes

$$\begin{array}{rcl} \|x_{n+1}-\rho\|^2 & \leq & \|x_n-\rho\|^2 - 2\alpha_n\Phi(2(\Phi^{-1}(a_0)) + \alpha_n\Phi(2(\Phi^{-1}(a_0)) \\ & \leq & \|x_n-\rho\|^2 - \alpha_n\Phi(2(\Phi^{-1}(a_0)) \\ & \leq & \|x_n-\rho\|^2 \\ & \leq & (2(\Phi^{-1}(a_0)))^2, \end{array} \tag{2.9}$$

which is a contradiction. Hence $\{x_n\}$ is a bounded sequence. So $\{y_n\}, \{z_n\}$, $\{T_1^n y_n\}, \{T_2^n z_n\}, \{T_3^n x_n\}$ are all bounded sequences.
So from (2.8),

$$\begin{array}{rcl} \|x_{n+1}-\rho\|^2 & \leq & \|x_n-\rho\|^2 - 2\alpha_n\Phi(2(\Phi^{-1}(a_0)) + 8\alpha_n(k_n-1)(\Phi^{-1}(a_0))^2 \\ & & +4\alpha_n\beta_n L((2+3L)(\Phi^{-1}(a_0))^2 \\ & & +4\alpha_n^2(1+L)(2+3L)(\Phi^{-1}(a_0))^2 \\ & = & \|x_n-\rho\|^2 - 2\alpha_n\Phi(2(\Phi^{-1}(a_0)) \\ & & +4\alpha_n(\Phi^{-1}(a_0))^2(2(k_n-1) \\ & & +(2+3L)(\beta_n L+\alpha_n(1+L))). \end{array} \tag{2.10}$$

Let $a_n = \|x_n-\rho\|$, $b_n = 2\alpha_n$,
$o(b_n) = 4\alpha_n(\Phi^{-1}(a_0))^2(2(k_n-1)+(2+3L)(\beta_n L+\alpha_n(1+L)))$. Then, we get that

$$a_{n+1}^2 \leq a_n^2 + o(b_n) - b_n\Phi(a_{n+1}), \quad \forall n \geq N_0.$$

Applying Lemma 1.10, we have that $a_n \to 0$ as $n \to \infty$. This completes the proof.
Remark 2.2 Our Theorem 2.1 answered the question posed by Rafiq et al. [9] in the affirmative, and the proof course of Theorem 2.1 is quite detailed than that of [9, Theorem 2.3].

3 Open Problem

It is natural to ask if the result of Theorem 2.1 can be extended to the case of finite family of mappings. Can we construct a sequence $\{x_n\}$ that will converge strongly to the common fixed point of such maps?

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