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Local existence and uniqueness of solutions for non stationary compressible fluid

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Abstract

This work is devoted to the study of a compressible viscoelastic fluids satisfying the Oldroyd-B model in a regular bounded domain. We prove the local existence of solutions and uniqueness of flows by a classical fixed point argument.

Keywords: *Local existence, viscoelastic fluids, Oldroyd-B model.*

1 Introduction

In this paper, we study the local existence of solutions for compressible viscoelastic fluid flows in the case of the Oldroyd-B model in a regular bounded domain in $\mathbb{R}^3$. We also show the uniqueness of solutions. We prove l'existence by using the classical method based on the Schauder fixed-point theorem. Valli, in [8], show the local existence in the case of the Navier-Stokes equations. The case of the Oldroyd model for incompressible fluid is studied by Guillop and Saut in [3]. Talhouk shows the existence and the uniqueness for Jeffreys model's in [6].

This paper is organized as follows. Section 2 is devoted to the modeling of the problem and to the definition of well-prepared initial conditions. The principal notation and results are detailed in Section 3. The local existence of regular solutions is given in Section 4.

2 The Modeling

2.1 Unsteady Flows of Compressible Viscoelastic Fluids

Consider unsteady flows of viscoelastic fluids in a bounded domain Ω^* of $\mathbb{R}^3$ with a regular boundary Γ^*. The system, obtained from the laws of conservation of momentum, and of mass, and from the constitutive equation of the fluid, reads as follows [4]: in $\mathrm{Q}^*_{\mathrm{T}^*} = (0, T^*) \times \Omega^*$,

$$\left\{\begin{array}{rcl} \rho^* \left(\dfrac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^* \right) & = & \rho^* \mathbf{f}^* + \mathbf{div}^*(\tau^* - p^* \mathbf{I}), \\ \dfrac{\partial \rho^*}{\partial t^*} + \mathrm{div}^*(\rho^* \mathbf{u}^*) & = & 0, \\ \tau^* + \lambda \dfrac{\mathcal{D}_a \tau^*}{\mathcal{D} t^*} & = & 2\eta \left(\mathbf{D}^* + \mu \dfrac{\mathcal{D}_a \mathbf{D}^*}{\mathcal{D} t^*} \right). \end{array}\right. \tag{1}$$

The *-variables are the dimensional ones in the domain of the flow Ω^*, and $T^* > 0$ is a dimensional time. The unknowns are the velocities $\mathbf{u}^*$, the density ρ^*, and the symmetric tensor of constraints τ^*. η is the total viscosity of the fluid, $\lambda > 0$ is the relaxation time, and μ is the retardation time $(0 < \mu < \lambda)$.
$\dfrac{\mathcal{D}_a \tau^*}{\mathcal{D} t^*}$ is an objective derivative of the tensor τ^*, given by

$$\frac{\mathcal{D}_a \tau^*}{\mathcal{D} t^*} = \left(\frac{\partial}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*) \right) \tau^* + \tau^* \mathbf{W}^* - \mathbf{W}^* \tau^* - a(\mathbf{D}^* \tau^* + \tau^* \mathbf{D}^*),$$

where $\mathbf{W}^* = \mathbf{W}^*[\mathbf{u}^*] = \dfrac{1}{2}(\nabla^* \mathbf{u}^* - \nabla^{*\mathrm{T}} \mathbf{u}^*)$ and $\mathbf{D}^* = \mathbf{D}^*[\mathbf{u}^*] = \dfrac{1}{2}(\nabla^* \mathbf{u}^* + \nabla^{*\mathrm{T}} \mathbf{u}^*)$ are, respectively, the rate of rotation and the rate of deformation tensors. a is a real parameter in $[-1, 1]$.

System (1) is completed by a condition on the boundary,

$$\mathbf{u}^* = 0 \text{ on } \Sigma^*_{\mathrm{T}^*} = (0, \mathrm{T}^*) \times \Gamma^*,$$

and by the initial data

$$\mathbf{u}^*(0, \cdot) = \mathbf{u}^*_0, \quad \rho^*(0, \cdot) = \rho^*_0, \quad \tau^*(0, \cdot) = \tau^*_0, \quad \text{in } \Omega^*.$$

We split τ^* into two parts: the Newtonian one τ^*_s related to the solvent, and the polymeric one τ^*_p. We may write

$$\tau^* = \tau^*_s + \tau^*_p = 2\eta_s \mathbf{D}^* + \tau^*_e,$$

where $\tau^*_e = \tau^*_p - \left(\frac{2\xi_s}{3} \mathrm{div}^* \mathbf{u}^*\right) \mathbf{I}$, and $\mathbf{I}$ is the identity tensor. $\eta_s = \eta\mu/\lambda$ and ξ_s are the solvent viscosity and the group viscosity, respectively. Since we are interested in a model for weakly compressible fluids, we suppose that $\xi_s = 0$. From the third equation in (1), we can deduce that τ_e^* satisfies the equation

$$\tau_e^* + \lambda \frac{\mathcal{D}_a \tau_e^*}{\mathcal{D}t^*} = 2\eta_e \mathbf{D}^*,$$

where $\eta_e = \eta - \eta_s$ is called the polymer viscosity. η_s and η_e are two non-negative numbers.

Therefore, under the assumption $\xi_s = 0$, System (1) is equivalent to the system in $\mathrm{Q}^*_{\mathrm{T}^*}$,

$$\left\{\begin{array}{rcl} \rho^*\left(\dfrac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^*\right) & = & \rho^*\mathbf{f}^* + \eta_s(\Delta^*\mathbf{u}^* + \nabla^*\mathrm{div}^*\mathbf{u}^*) - \nabla^* p^* + \mathbf{div}^*\tau^*, \\ \dfrac{\partial \rho^*}{\partial t^*} + \mathrm{div}^*(\rho^*\mathbf{u}^*) & = & 0, \\ \tau^* + \lambda \dfrac{\mathcal{D}_a \tau^*}{\mathcal{D}t^*} & = & 2\eta_e \mathbf{D}^*[\mathbf{u}^*], \end{array}\right. \tag{2}$$

where we have denoted τ_e^* by τ^* to simplify the notation.

2.2 Well-Prepared Initial Conditions

We first define the Mach number ε as being the ratio of the typical velocity of the fluid U_0 to the speed of sound $\left(\frac{dp^*}{d\rho^*}(\overline{\rho}_0^*)\right)^{1/2}$ in the same fluid at the same state. We divide the density $\rho^* = \rho^{*\varepsilon}$ into two parts: a *constant* one $\overline{\rho}_0^*$, independent of ε, and a remainder, which is small for small $\varepsilon' s$, say

$$\rho^{*\varepsilon} = \overline{\rho}_0^* + \mathcal{O}(\varepsilon^2) = \overline{\rho}_0^* + \varepsilon^2 \sigma^{*\varepsilon}.$$

We also suppose that the initial conditions $\rho_0^{*\varepsilon}$, $\mathbf{u}_0^{*\varepsilon}$ and $\tau_0^{*\varepsilon}$ are *well-prepared*, which means that they take a similar form, say

$$\begin{array}{rcl} \rho_0^{*\varepsilon} & = & \overline{\rho}_0^* + \mathcal{O}(\varepsilon^2) = \overline{\rho}_0^* + \varepsilon^2 \sigma_0^{*\varepsilon}, \\ \mathbf{u}_0^{*\varepsilon} & = & \mathbf{v}_0^* + \mathbf{v}_0^{*\varepsilon}, \text{ with } \mathrm{div}\, \mathbf{v}_0^* = 0, \\ \tau_0^{*\varepsilon} & = & \mathbf{S}_0^* + \mathbf{S}_0^{*\varepsilon}, \end{array}$$

where $\mathbf{v}_0^*$ and $\mathbf{S}_0^*$ are, respectively, a vector and a symmetric tensor, both independent of ε.

We assume

$$\mathfrak{m}^* = \min_{\overline{\Omega}^*} \rho_0^* > 0 \quad \text{and} \quad \mathfrak{M}^* = \max_{\overline{\Omega}^*} \rho_0^*.$$

Assuming that $p^* = p^*(\rho^*)$ is regular, say class C^3 at least, we remark

$$\frac{dp^*}{d\rho^*}(\overline{\rho}_0^* + \varepsilon^2\sigma^*) - \frac{dp^*}{d\rho^*}(\overline{\rho}_0^*) = \varepsilon^2 \int_0^1 \frac{d^2 p^*}{d\rho^{*2}}(\overline{\rho}_0^* + s\varepsilon^2\sigma^*)\, ds.$$

We introduce the function w^*, defined by $w^*(\sigma^*) = \frac{dp^*}{d\rho^*}(\overline{\rho}_0^* + \varepsilon^2\sigma^*) - \frac{dp^*}{d\rho^*}(\overline{\rho}_0^*)$, and remark that w^* depends on ε, satisfies $w^*(0) = 0$, and is of class C^2 at least.

Replacing ρ^* by its value in the first equation (2), one infers

$$(\overline{\rho}_0^* + \varepsilon^2\sigma^*)\left(\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^*\right) + \varepsilon^2\frac{dp^*}{d\rho^*}(\overline{\rho}_0^* + \varepsilon^2\sigma^*)\nabla^*\sigma^*$$
$$= (\overline{\rho}_0^* + \varepsilon^2\sigma^*)\mathbf{f}^* + \eta_s(\Delta^*\mathbf{u}^* + \nabla^*\mathrm{div}^*\mathbf{u}^*) + \mathbf{div}^*\tau^*.$$

We can also rewrite this equality, by taking into account the definitions of $w^*(\sigma^*)$ and of the Mach number ε, in the form

$$(\overline{\rho}_0^* + \varepsilon^2\sigma^*)\left(\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^*\right) + (\mathrm{U}_0)^2\nabla^*\sigma^*$$
$$= (\overline{\rho}_0^* + \varepsilon^2\sigma^*)\mathbf{f}^* + \eta_s(\Delta^*\mathbf{u}^* + \nabla^*\mathrm{div}^*\mathbf{u}^*) + \mathbf{div}^*\tau^* - \varepsilon^2 w^*(\sigma^*)\nabla^*\sigma^*.$$

From the second equation in (2) we easily deduce

$$\varepsilon^2\frac{\partial\sigma^*}{\partial t^*} + \overline{\rho}_0^*\mathrm{div}^*\mathbf{u}^* + \varepsilon^2\mathrm{div}^*(\sigma^*\mathbf{u}^*) = 0.$$

Finally, System (2) can be written as follows, in $\mathrm{Q}^*_{\mathrm{T}^*}$,

$$\begin{cases} (\overline{\rho}_0^* + \varepsilon^2\sigma^*)\left(\dfrac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^*\right) + (\mathrm{U}_0)^2\nabla^*\sigma^* = (\overline{\rho}_0^* + \varepsilon^2\sigma^*)\mathbf{f}^* + \mathbf{div}^*\tau^* \\ \qquad + \eta_s(\Delta^*\mathbf{u}^* + \nabla^*\mathrm{div}^*\mathbf{u}^*) - \varepsilon^2 w^*(\sigma^*)\nabla^*\sigma^*, \\ \varepsilon^2\dfrac{\partial\sigma^*}{\partial t^*} + \overline{\rho}_0^*\mathrm{div}^*\mathbf{u}^* + \varepsilon^2\mathrm{div}^*(\sigma^*\mathbf{u}^*) = 0, \\ \tau^* + \lambda\dfrac{\mathcal{D}_a\tau^*}{\mathcal{D}t^*} = 2\eta_e\mathbf{D}^*[\mathbf{u}^*]. \end{cases} \tag{3}$$

2.3 Dimensionless Variables

We introduce the dimensionless variables,

$$\mathbf{x}^* = L_0\mathbf{x}, \quad t^* = \frac{L_0}{\mathrm{U}_0}t, \quad \rho^* = a_0\rho, \quad w^*(\sigma^*) = (\mathrm{U}_0)^2 w(\sigma),$$

$$\mathbf{u}^* = \mathrm{U}_0\mathbf{u}, \quad \sigma^* = a_0\sigma, \quad \tau^* = T_0\tau, \quad p^*(\rho^*) = T_0 p(\rho), \quad \mathbf{f}^* = \frac{(\mathrm{U}_0)^2}{L_0}\mathbf{f},$$

where L_0 represents a typical length of the flow. The real numbers $a_0 = \frac{\eta}{\mathrm{U}_0 L_0}$ and $T_0 = \frac{\eta \mathrm{U}_0}{L_0}$ characterize the density and the stress tensor of the fluid. Ω denotes the non-dimensional domain of the flow, with boundary Γ, and $T > 0$ a non-dimensional time.

We introduce three non-dimensional numbers: a number α similar to the Reynolds number for incompressible flows, the Weissenberg number We, and a number ω relative to the viscosities of the fluid,

$$\alpha = \frac{\overline{\rho}_0^*}{a_0} = \frac{\overline{\rho}_0^* \mathrm{U}_0 L_0}{\eta}, \quad \mathrm{We} = \frac{\lambda \mathrm{U}_0}{L_0}, \quad \omega = 1 - \frac{\eta_s}{\eta}.$$

We also define

$$w(\sigma) = \alpha \left\{ \frac{dp}{d\rho}(\alpha + \varepsilon^2\sigma) - \frac{dp}{d\rho}(\alpha) \right\}.$$

In dimensionless variables, System (3) takes the form, in $\mathrm{Q_T} = (0,T)\times\Omega$,

$$\left\{ \begin{array}{rcl} \mathbf{u}' + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{1}{\alpha+\varepsilon^2\sigma}\nabla\sigma & = & \mathbf{f} + \dfrac{1-\omega}{\alpha+\varepsilon^2\sigma}(\Delta\mathbf{u} + \nabla \mathrm{div}\,\mathbf{u}) + \dfrac{\mathbf{div}\,\tau}{\alpha+\varepsilon^2\sigma} \\ & & -\dfrac{\varepsilon^2 w(\sigma)\nabla\sigma}{\alpha+\varepsilon^2\sigma}, \\ \sigma' + \varepsilon^{-2}\alpha\,\mathrm{div}\,\mathbf{u} + \mathrm{div}\,(\sigma\mathbf{u}) & = & 0, \\ \tau + \mathrm{We}\{\tau' + (\mathbf{u}\cdot\nabla)\tau + \mathbf{g}(\nabla\mathbf{u},\tau)\} & = & 2\omega\mathbf{D}[\mathbf{u}], \end{array} \right. \tag{4}$$

with the notation $\mathbf{u}' = \dfrac{\partial\mathbf{u}}{\partial t}$, $\sigma' = \dfrac{\partial\sigma}{\partial t}$ and $\tau' = \dfrac{\partial\tau}{\partial t}$, and

$$\mathbf{g}(\nabla\mathbf{u},\tau) = \tau\mathbf{W}[\mathbf{u}] - \mathbf{W}[\mathbf{u}]\tau - a\Big(\mathbf{D}[\mathbf{u}]\tau + \tau\mathbf{D}[\mathbf{u}]\Big).$$

Introducing the differential operator $A = -(\Delta + \nabla\mathrm{div}\,)$ we may rewrite System (4) as follows, in $\mathrm{Q_T}$,

$$\left\{ \begin{array}{rcl} \alpha\Big(\mathbf{u}' + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) + (1-\omega)A\mathbf{u} + \nabla\sigma & = & \mathbf{F}(\mathbf{u},\sigma,\tau) + \mathbf{div}\,\tau, \\ \sigma' + (\mathbf{u}\cdot\nabla)\sigma + \sigma\mathrm{div}\,\mathbf{u} & = & -\varepsilon^2\alpha\,\mathrm{div}\,\mathbf{u}, \\ \tau + \mathrm{We}\Big(\tau' + (\mathbf{u}\cdot\nabla)\tau + \mathbf{g}(\nabla\mathbf{u},\tau)\Big) & = & 2\omega\mathbf{D}[\mathbf{u}], \end{array} \right. \tag{5}$$

with

$$\mathbf{F}(\mathbf{u},\sigma,\tau) = \alpha\mathbf{f} + \frac{(1-\omega)\varepsilon^2\sigma}{\alpha+\varepsilon^2\sigma}A\mathbf{u} + \frac{\varepsilon^2(\sigma - w(\sigma))}{\alpha+\varepsilon^2\sigma}\nabla\sigma - \frac{\varepsilon^2\sigma}{\alpha+\varepsilon^2\sigma}\mathbf{div}\,\tau. \tag{6}$$

System (5) is completed by an homogeneous condition on the boundary,

$$\mathbf{u} = 0 \text{ on } \Sigma_{\mathrm{T}} = (0,\mathrm{T})\times\Gamma, \tag{7}$$

and by three initial conditions,

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0, \quad \sigma(0,\cdot) = \sigma_0, \quad \tau(0,\cdot) = \tau_0, \quad \text{in } \Omega. \tag{8}$$

We also assume the followings,

$$0 < \mathfrak{m}_1 = \frac{\mathfrak{m}^*}{a_0} \leq \alpha + \varepsilon^2\sigma_0 \leq \mathfrak{M}_1 = \frac{\mathfrak{M}^*}{a_0}, \quad \text{in } \Omega,$$

where $\mathfrak{m}_1$ and $\mathfrak{M}_1$ are some given constants.

3 The Notation and Main Results

3.1 Notation

Ω is a bounded domain in $\mathbb{R}^3$, with a regular boundary Γ, and $\mathbf{n}$ denotes the unit outward-pointing normal vector to Γ. For $\mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$, we denote by $|\mathbf{x}|$ its Euclidean norm.

We will use the following spaces: the Lebesgue spaces $\mathrm{L}^p(\Omega)$, $1 \leq p \leq +\infty$, with norms $||\cdot||_{\mathrm{L}^p}$ (except for the $\mathrm{L}^2(\Omega)$-norm, which is denoted by $||\cdot||$); the Sobolev space $\mathrm{H}^k(\Omega)$, $k \in \mathbb{N}^*$, with norm $||\cdot||_k$ and inner product $((\cdot,\cdot))_k$; the vector spaces $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^k(\Omega)$ of vector-valued or tensor-valued functions with components in $\mathrm{L}^2(\Omega)$ and $\mathrm{H}^k(\Omega)$ respectively, their norms being denoted in the same way as above. We will also use the homogeneous Sobolev space $\mathrm{H}_0^1(\Omega)$ and its dual $\mathrm{H}^{-1}(\Omega)$.

If I is an interval of $\mathbb{R}_+$ and $k \in \mathbb{N}$, $\mathrm{C}(\overline{I}; \mathbf{H}^k(\Omega))$ is the space of vector- or tensor-valued functions which are continuous on $\overline{I}$ with values in $\mathbf{H}^k(\Omega)$. The norm, in this space, is denoted by $||\cdot||_{\mathcal{C},k}$. $\mathrm{C}_b(\overline{I}; \mathbf{H}^k(\Omega))$ is the space of functions of $\mathrm{C}(I; \mathbf{H}^k(\Omega))$ which are bounded on $\overline{I}$.

The space $\mathrm{L}^p(I; \mathbf{H}^k(\Omega))$, for $1 \leq p \leq +\infty$, and $k \in \mathbb{N}$, consists of p-integrable functions on I with values in $\mathbf{H}^k(\Omega)$. For $1 \leq p \leq +\infty$, $k \in \mathbb{N}$ and $0 < \mathrm{T} \leq \infty$, the norm in $\mathrm{L}^p((0,\mathrm{T}), \mathbf{H}^k(\Omega))$ is denoted by $[\,\cdot\,]_{p,k,\mathrm{T}}$. $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathbf{H}^k(\Omega))$ is the set of functions which are in $\mathrm{L}^2(I; \mathbf{H}^k(\Omega))$ for all bounded interval I in $\mathbb{R}_+$.

The letters C, c_i or c_i^j, $i, j = 1, 2, \cdots$, will denote constants taking different values, but not depending on ε. C_Ω will be a constant, taking different values, and depending only on Ω. $(2.1)_n$ denotes the $n-$th equation of System (2.1).

3.2 The Main Result

Recall the problem under study:

$$\begin{cases} \alpha\Big(\mathbf{u}' + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) + (1-\omega)A\mathbf{u} + \nabla\sigma &= \mathbf{F}(\mathbf{u},\sigma,\tau) + \mathbf{div}\,\tau, \\ \sigma' + (\mathbf{u}\cdot\nabla)\sigma + \sigma\mathrm{div}\,\mathbf{u} &= -\varepsilon^2\alpha\,\mathrm{div}\,\mathbf{u}, \\ \tau + \mathrm{We}\Big(\tau' + (\mathbf{u}\cdot\nabla)\tau + \mathbf{g}(\nabla\mathbf{u},\tau)\Big) &= 2\omega\mathbf{D}[\mathbf{u}], \quad \text{in } \mathrm{Q_T}, \\ \mathbf{u}(0,\cdot) &= \mathbf{u}_0, \qquad \text{in } \Omega, \\ \sigma(0,\cdot) &= \sigma_0, \qquad \text{in } \Omega, \\ \tau(0,\cdot) &= \tau_0, \qquad \text{in } \Omega, \\ \mathbf{u} &= 0, \qquad \text{on } \Sigma_\mathrm{T}, \end{cases} \tag{9}$$

where $\mathbf{F}$ is defined by (6).

Theorem 3.1 (Existence of a local solution) *Assume $\Omega \subset \mathbb{R}^3$ is a domain of class C^3. Let $\mathfrak{m}_1$ and $\mathfrak{M}_1$ be two real constants such that $0 < \mathfrak{m}_1 \le \mathfrak{M}_1$. Assume*

$$\mathbf{f} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}^1(\Omega)), \ \textit{with}\, \mathbf{f}' \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}^{-1}(\Omega)),\ \mathbf{u}_0 \in \mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega),\ \tau_0 \in \mathbf{H}^2(\Omega),$$

$$\sigma_0 \in \mathrm{H}^2(\Omega),\ \textit{with} \int_\Omega \sigma_0(\mathbf{x})d\mathbf{x} = 0,\ \textit{and}\ 0 < \mathfrak{m}_1 \le \alpha + \varepsilon^2\sigma_0 \le \mathfrak{M}_1\,,\ \textit{in}\ \Omega.$$

Then there exists a time $\mathrm{T}_1 > 0$ and a solution $(\mathbf{u},\sigma,\tau)$ of Problem (5)-(8) in $Q_{\mathrm{T}_1} = (0,\mathrm{T}_1)\times\Omega$, satisfying

$$\begin{aligned} &\mathbf{u} \in \mathrm{L}^2(0,\mathrm{T}_1;\mathbf{H}^3(\Omega)) \cap \mathrm{C}([0,\mathrm{T}_1];\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega)), \\ &\mathbf{u}' \in \mathrm{L}^2(0,\mathrm{T}_1;\mathbf{H}^1_0(\Omega)) \cap \mathrm{C}([0,\mathrm{T}_1];\mathbf{L}^2(\Omega)), \\ &(\tau,\sigma) \in \mathrm{C}([0,\mathrm{T}_1];\mathbf{H}^2(\Omega)\times\mathrm{H}^2(\Omega)), \quad (\tau',\sigma') \in \mathrm{C}([0,\mathrm{T}_1];\mathbf{H}^1(\Omega)\times\mathrm{H}^1(\Omega)), \end{aligned}$$

with

$$\int_\Omega \sigma(\cdot,\mathrm{x})\mathrm{dx} = 0, \quad \textit{in}\ [0,\mathrm{T}_1],\ \textit{and}\ \ \frac{\mathfrak{m}_1}{2} \le \alpha + \varepsilon^2\sigma \le 2\mathfrak{M}_1\,, \quad \textit{in}\ \overline{Q}_{\mathrm{T}_1}.$$

Theorem 3.2 (Uniqueness of a local solution) *There exist a unique solution of Problem (5)-(8), given in Theorem 3.1.*

To show that the local solution found in Theorem 3.1 exists for all times under certain regularity and smallness conditions on the data, we also assume that the function $w \in \mathrm{C}^2(\mathbb{R})$ has the following properties: for all $h \in \mathrm{L}^2(0,\mathrm{T};\mathrm{H}^2(\Omega))$,

$$\begin{aligned} ||(w(h))'|| \le C\,||h'||\,, \qquad ||w(h)|| \le C\,||h||\,, \\ ||w(h)||_k \le C\,||h||_k\,, \quad k = 1,2, \end{aligned} \tag{10}$$

for some constant C depending on Ω and w.

Remark 3.3 *There are several examples of functions $p = p(\rho)$, for which w satisfies the conditions above. Let us quote the case where the pressure is given by the linear state law $p(\rho) = \frac{1}{\varepsilon^2}(\rho - \alpha)$, as well as the case of isothermal compressible perfect fluids, where $p(\rho) = (C_s)^2\rho$, and C_s is the velocity of sound in the fluid.*

4 Existence and Uniqueness of Local Solutions

We prove Theorem 3.1 by using the classical method based on the Schauder fixed-point theorem. To do that in our case, we study three linear problems: the first one has the velocity $\mathbf{u}$ as unknown, and the next ones are two transport equations for the density σ and for the stress tensor τ respectively. The parameter ε is fixed in the interval $(0, 1]$.

Let $\mathbf{w}$, π and ψ a given vector, function and the symmetric tensor of constraints respectively. Let T a positive real number, $Q_{\mathrm{T}} = \Omega\times]0, \mathrm{T}[$ and $\Sigma_{\mathrm{T}} = \partial\Omega\times]0, \mathrm{T}[$. Consider the linear problem,

$$\left\{\begin{array}{rcll} \alpha\mathbf{u}' + (1-\omega)A\mathbf{u} & = & \mathfrak{F}, & \\ \sigma' + (\mathbf{w}.\nabla)\sigma + \sigma\mathrm{div}\,\mathbf{w} & = & \mathcal{G}, & \\ \tau + \mathrm{We}\{\tau' + (\mathbf{w}.\nabla)\tau + \mathbf{g}(\nabla\mathbf{w}, \tau)\} & = & 2\omega\mathbf{D}[\mathbf{w}], & \text{in } \mathrm{Q_T}, \\ \mathbf{u}(0,x) & = & \mathbf{u}_0(x), & \\ \sigma(0,x) & = & \sigma_0(x), & \\ \tau(0,x) & = & \tau_0(x), & \text{in } \Omega, \\ \mathbf{u} & = & 0, & \text{on } \Sigma_{\mathrm{T}}, \end{array}\right. \tag{11}$$

with

$$\mathfrak{F} = \mathbf{F}(\mathbf{w}, \pi, \psi) - \alpha(\mathbf{w}.\nabla)\mathbf{w} - \nabla\pi + \mathbf{div}\,\psi, \tag{12}$$

$$\mathcal{G} = -\varepsilon^{-2}\mathrm{div}\,\mathbf{w}. \tag{13}$$

and

$$\frac{\mathfrak{m}_1}{2} \leq \alpha + \varepsilon^2\pi \leq 2\mathfrak{M}_1, \quad \text{in } \overline{Q}_{\mathrm{T}}. \tag{14}$$

4.1 Linear problem concerning the velocity u

Consider the linear problem concerning the velocity $\mathbf{u}$,

$$\left\{\begin{array}{rcll} \alpha\mathbf{u}' + (1-\omega)A\mathbf{u} & = & \mathfrak{F}, & \text{in } \mathrm{Q_T}, \\ \mathbf{u}(0,x) & = & \mathbf{u}_0(x), & \text{in } \Omega, \\ \mathbf{u} & = & 0, & \text{on } \Sigma_{\mathrm{T}}, \end{array}\right. \tag{15}$$

where $A\mathbf{u} = -(\Delta\mathbf{u} + \nabla\mathrm{div}\,\mathbf{u})$, $\mathfrak{F}$ and $\mathbf{u}_0$ are given and $0 < \mathrm{T} \leq +\infty$.

The first Lemma concerns the existence of a unique solution of (15). By classical result of Agmon-Douglis-Nirenberg [1], $A = -\Delta - \nabla\mathrm{div}$ is a strongly elliptic operator, and generates an analytic semigroup in $\mathbf{L}^2(\Omega)$ with domain $D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega)$ (we can see for instance [8]).

Lemma 4.1 *Let* $\Omega \subset \mathbb{R}^3$ *of class* C^2, $\mathfrak{F} \in \mathrm{L}^2(0,\mathrm{T};\mathbf{L}^2(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{H}^1_0(\Omega)$. *Then there exists a unique solution of problem (15)*

$$\begin{aligned} \mathbf{u} &\in \mathrm{L}^2(0,\mathrm{T};\mathbf{H}^2(\Omega)) \cap \mathrm{C}([0,\mathrm{T}];\mathbf{H}^1_0(\Omega)), \\ \mathbf{u}' &\in \mathrm{L}^2(0,\mathrm{T};\mathbf{L}^2(\Omega)). \end{aligned}$$

Moreover, this solution satisfies the estimate

$$\begin{aligned} &\frac{\alpha}{2}||\mathbf{u}'||^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{L}^2(\Omega))} + \frac{(1-\omega)^2}{2}||A\mathbf{u}||^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{L}^2(\Omega))} + (1-\omega)||D\mathbf{u}||^2_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{L}^2(\Omega))} \\ &\quad +(1-\omega)||\mathrm{div}\,\mathbf{u}||^2_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{L}^2(\Omega))} \leq 4(1-\omega)\,||D\mathbf{u}_0||^2 + ||\mathfrak{F}||^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{L}^2(\Omega))}. \qquad (16) \end{aligned}$$

Proof.
By classical result of Agmon-Douglis-Nirenberg [1], $A = -\Delta - \nabla\mathrm{div}$ is a strongly elliptic operator, and generates an analytic semigroup in $\mathbf{L}^2(\Omega)$ with domain $D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega)$ (we can see for instance [8]).
We start by showing the estimate (16). Multiply $(15)_1$ in $\mathbf{L}^2(\Omega)$ by $\mathbf{u}' + \alpha(1-\omega)A\mathbf{u}$, then

$$\begin{aligned} \int_\Omega |\mathbf{u}'|^2 + 2(1-\omega)\int_\Omega \mathbf{u}' \cdot A\mathbf{u} + (1-\omega)^2\int_\Omega |A\mathbf{u}|^2 \\ = \int_\Omega \mathfrak{F}\cdot\mathbf{u}' + (1-\omega)\int_\Omega \mathfrak{F}\cdot A\mathbf{u}. \end{aligned}$$

Integrate by parts the second term, we obtain

$$\begin{aligned} ||\mathbf{u}'||^2 + (1-\omega)\frac{\mathrm{d}}{\mathrm{dt}}\Big(\,||D\mathbf{u}||^2 + ||\mathrm{div}\,\mathbf{u}||^2\Big) + (1-\omega)^2\,||A\mathbf{u}||^2 \\ \leq ||\mathfrak{F}||\cdot||\mathbf{u}'|| + \frac{(1-\omega)}{4}\,||\mathfrak{F}||\cdot||A\mathbf{u}||\,. \end{aligned}$$

On the other hand, using Young's inequality on the two terms right, we get

$$\begin{aligned} ||\mathfrak{F}||\cdot||\mathbf{u}'|| &\leq \frac{1}{2}\,||\mathfrak{F}||^2 + \frac{1}{2}\,||\mathbf{u}'||^2, \\ (1-\omega)\,||\mathfrak{F}||\cdot||A\mathbf{u}|| &\leq \frac{1}{2}\,||\mathfrak{F}||^2 + \frac{(1-\omega)^2}{2}\,||A\mathbf{u}||^2. \end{aligned}$$

Integrate over $[0,\mathrm{T}]$ and use the inequality

$$||\mathrm{div}\,\mathbf{u}_0|| \leq 3\,||D\mathbf{u}_0||\,,$$

then we get (16). □

The second Lemma give some stronger estimates.

Lemma 4.2 ([8, 6]) *Under the conditions of Lemma 4.1 and if* $\partial\Omega \in \mathrm{C}^3$, $\mathfrak{F} \in \mathrm{L}^2(0,\mathrm{T};\mathbf{H}^1(\Omega))$, $\mathfrak{F}' \in \mathrm{L}^2(0,\mathrm{T};\mathbf{H}^{-1}(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{H}^1(\Omega) \cap \mathbf{H}^1_0(\Omega)$. *Then the solution* $\mathbf{u}$ *of problem (15) given by Lemma 4.1 is such that*

$$\begin{aligned} \mathbf{u} &\in \mathrm{L}^2(0,\mathrm{T};\mathbf{H}^3(\Omega)) \cap \mathrm{C}([0,\mathrm{T}];\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega)), \\ \mathbf{u}' &\in \mathrm{L}^2(0,\mathrm{T};\mathbf{H}^1_0(\Omega)) \cap \mathrm{C}([0,\mathrm{T}];\mathbf{L}^2(\Omega)). \end{aligned}$$

and there exists a constant C_1, *depend only in* Ω, T, α *and* ω, *such that one has the estimate*

$$\begin{aligned} &||\mathbf{u}||^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{H}^3(\Omega))} + ||\mathbf{u}||^2_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^2(\Omega)\cap\mathrm{H}^1_0(\Omega))} + ||\mathbf{u}'||^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{H}^1_0(\Omega))} + ||\mathbf{u}'||^2_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{L}^2(\Omega))} \\ &\qquad \leq C_1\{||A\mathbf{u}_0||^2 + ||\mathfrak{F}(0)||^2 + ||\mathfrak{F}||^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{H}^1(\Omega))} + ||\mathfrak{F}'||^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{H}^{-1}(\Omega))}\}. \end{aligned} \tag{17}$$

Proof.
Derive in terms of t the equation $(15)_1$, then we obtain

$$\mathbf{u}'' + \alpha(1-\omega)A\mathbf{u}' = \mathfrak{F}', \qquad \text{in } \mathrm{Q_T},$$

and $\mathbf{u}'_{|\partial\Omega}(t) = 0$ for all $t \in [0,\mathrm{T}]$. Let $\mathbf{v} = \mathbf{u}'$, then $\mathbf{v}$ verify the system

$$\left\{ \begin{aligned} \mathbf{v}' + \alpha(1-\omega)A\mathbf{v} &= \mathfrak{F}', && \text{in } \mathrm{Q_T}, \\ \mathbf{v}(0) &= \mathbf{v}_0 = \mathfrak{F}(0) - \alpha(1-\omega)A\mathbf{u}_0, && \text{in } \Omega, \\ \mathbf{v} &= 0, && \text{on } \Sigma_\mathrm{T}. \end{aligned} \right. \tag{18}$$

Multiply by $\mathbf{v}$ the equation $(20)_1$ and integrate on Ω. It comes

$$\int_\Omega \mathbf{v}' \cdot \mathbf{v} + \alpha(1-\omega)\int_\Omega A\mathbf{v}\cdot\mathbf{v} = <\mathfrak{F}', \mathbf{v}>_{\mathbf{H}^{-1},\mathbf{H}^1_0}.$$

After calculation, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}||\mathbf{v}||^2 + \frac{3\alpha(1-\omega)}{4}||D\mathbf{v}||^2 + \alpha(1-\omega)\,||\mathrm{div}\,\mathbf{v}||^2 \leq \frac{1}{\alpha(1-\omega)}||\mathfrak{F}'||^2_{\mathbf{H}^{-1}}.$$

Integrate on $[0,\mathrm{T}]$ and replace $\mathbf{v}$ and $\mathbf{v}_0$ by their values

$$\begin{aligned} &\frac{1}{2}||\mathbf{u}'||^2_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{L}^2(\Omega))} + \frac{3\alpha(1-\omega)}{4}||D\mathbf{u}'||^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{L}^2(\Omega))} \\ &\quad + \alpha(1-\omega)\,||\mathrm{div}\,\mathbf{u}'||^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{L}^2(\Omega))} \leq \frac{1}{\alpha(1-\omega)}||\mathfrak{F}'||^2_{\mathrm{L}^2(0,\mathrm{T},\mathbf{H}^{-1}(\Omega))} \\ &\qquad + \frac{1}{2}\Big(||\mathfrak{F}(0)||^2 + \alpha(1-\omega)\,||A\mathbf{u}_0||^2\Big). \end{aligned} \tag{19}$$

Finally, inequality (17) follows from inequality (16) and (19). $\square$

4.2 Resolution of the Transport Problems

We consider the following two linear transport problems,

$$\begin{cases} \sigma' + (\mathbf{w}\cdot\nabla)\sigma + \sigma\,\mathrm{div}\,\mathbf{w} &= -\varepsilon^{-2}\alpha\mathrm{div}\,\mathbf{w}, \quad \text{in } \mathrm{Q_T}, \\ \sigma(0,\cdot) &= \sigma_0, \qquad\qquad\quad \text{in } \Omega, \end{cases} \tag{20}$$

and

$$\begin{cases} \tau + \mathrm{We}\Big(\tau' + (\mathbf{w}\cdot\nabla)\tau + \mathbf{g}(\nabla\mathbf{w},\tau)\Big) &= 2\omega\mathbf{D}[\mathbf{w}], \quad \text{in } \mathrm{Q_T}, \\ \tau(0,\cdot) &= \tau_0, \qquad\qquad \text{in } \Omega, \end{cases} \tag{21}$$

where σ_0 and τ_0 are, respectively, some given function and symmetric tensor defined in Ω. The existence of solutions to this problems follows from the classical method of characteristics. (see for example [3, 6, 8]). The lemmas below give some estimates of the solutions of these problems.

Lemma 4.3 ([8]) *Let* $\Gamma \in \mathrm{C}^1$*,*

$$\mathbf{w} \in \mathrm{L}^1(0,\mathrm{T},\mathbf{H}^3(\Omega)), \quad \mathbf{w}.\mathbf{n} = 0 \text{ on } \Sigma_\mathrm{T}, \text{ and } \sigma_0 \in \mathbf{H}^2(\Omega),$$

with $\displaystyle\int_\Omega \sigma_0\, d\boldsymbol{x} = 0$*. Then there exists a unique solution* $\sigma \in \mathrm{C}([0,\mathrm{T}];\mathrm{H}^2(\Omega))$ *of (20) such that*

$$\int_\Omega \sigma(\cdot,\mathbf{x})d\mathrm{x} = 0 \quad in \quad [0,\mathrm{T}],$$

and satisfying the following estimate

$$||\sigma||_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^2(\Omega))} \le (||\sigma_0|| + \alpha\varepsilon^{-2})\exp\left(C_\Omega\,||\mathbf{w}||_{\mathrm{L}^1(0,\mathrm{T},\mathrm{H}^3(\Omega))}\right),$$

for some positive constant C_Ω *depending on* Ω*.*
If, in addition,

$$\mathbf{w} \in \mathrm{C}([0,\mathrm{T}];\mathbf{H}^2(\Omega)),$$

then $\sigma' \in \mathrm{C}([0,\mathrm{T}];\mathrm{H}^1(\Omega))$ *satisfies*

$$||\sigma'||_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^1(\Omega))} \le C_\Omega\,||\mathbf{w}||_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^2(\Omega))}\,(||\sigma_0||+\alpha\varepsilon^{-2})\exp\Big(C_\Omega\,||\mathbf{w}||_{\mathrm{L}^1(0,\mathrm{T},\mathrm{H}^3(\Omega))}\Big).$$

Lemma 4.4 ([3]) *Let* $\Omega \subset \mathbb{R}^3$ *be a domain of class* C^3*,*

$$\mathbf{w} \in \mathrm{L}^1(0,\mathrm{T};\mathbf{H}^3(\Omega)\cap\mathbf{H}^1_0(\Omega)) \text{ and } \tau_0 \in \mathbf{H}^2(\Omega).$$

Then there exists a unique solution $\tau \in \mathrm{C}([0,\mathrm{T}];\mathbf{H}^2(\Omega))$ *of (21), such that*

$$||\tau||_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^2(\Omega))} \leq \Big(||\tau_0||^2 + \frac{2\omega}{C_\Omega \mathrm{We}}\Big) \exp\Big(C_\Omega \, ||\mathbf{w}||_{\mathrm{L}^1(0,\mathrm{T},\mathrm{H}^3(\Omega))} \Big),$$

for some positive constant C_Ω *depending on* Ω.
If, in addition,

$$\mathbf{w} \in \mathrm{C}([0,\mathrm{T}];\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega)),$$

then $\tau' \in \mathrm{C}([0,\mathrm{T}];\mathbf{H}^1(\Omega))$ *satisfies*

$$||\tau'||_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^1(\Omega))} \leq$$
$$C_0\Big(||\mathbf{w}||_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^2(\Omega))} + \frac{1}{C_\Omega \mathrm{We}}\Big)\Big(||\tau_0|| + \frac{2\omega}{C_\Omega \mathrm{We}}\Big) \exp\Big(C_\Omega \, ||\mathbf{w}||_{\mathrm{L}^1(0,\mathrm{T},\mathrm{H}^3(\Omega))} \Big).$$

4.3 Proof of Theorem 3.1

We are now in a position to prove the local existence of a solution to problem (9). We apply the Theorem of fixed-point of Schauder.

Take $\mathrm{T} > 0$, $\mathfrak{B}_1$, $\mathfrak{B}_2 > 0$, and define

$$\begin{aligned}
\mathfrak{R}_\mathrm{T} \;=\; & \{(\mathbf{w},\pi,\psi), \\
& \mathbf{w} \in \mathrm{C}([0,\mathrm{T}];\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega)) \cap \mathrm{L}^\infty(0,\mathrm{T},\mathbf{H}^3(\Omega)), \\
& \mathbf{w}' \in \mathrm{C}([0,\mathrm{T}];\mathbf{L}^2(\Omega)) \cap \mathrm{L}^2(0,\mathrm{T},\mathbf{H}^1_0(\Omega)), \\
& \pi \in \mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^2(\Omega)), \pi' \in \mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^1(\Omega)), \\
& \psi \in \mathrm{L}^\infty(0,\mathrm{T},\mathbf{H}^2(\Omega)), \psi' \in \mathrm{L}^\infty(0,\mathrm{T},\mathbf{H}^1(\Omega)), \\
& \mathbf{w}(0) = \mathbf{u}_0,\ \pi(0) = \sigma_0, \psi(0) = \tau_0 \text{ in } \Omega, \mathbf{w} = 0 \text{ in } \Sigma_\mathrm{T}, \\
& ||\mathbf{w}||^2_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^2(\Omega)\cap\mathrm{H}^1_0(\Omega))} + ||\mathbf{w}||^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{H}^3(\Omega))} + ||\mathbf{w}'||^2_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{L}^2(\Omega))} + \\
& ||\mathbf{w}'||^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{H}^1_0(\Omega))} \leq \mathfrak{B}_1, \\
& ||\pi||_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^2(\Omega))} + ||\psi||_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^2(\Omega))} \leq \mathfrak{B}_1, \\
& ||\pi'||_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^1(\Omega))} + ||\psi'||_{\mathrm{L}^\infty(0,\mathrm{T},\mathrm{H}^1(\Omega))} \leq \mathfrak{B}_2, \\
& \frac{\mathfrak{m}_1}{2} \leq \alpha + \varepsilon^2\pi(t,x) \leq 2\mathfrak{M}_1\,, \quad \text{ in } \overline{Q}_\mathrm{T}\}.
\end{aligned}$$

Choose $\mathfrak{B}_1$ such that

$$\mathfrak{B}_1 > \max\{C_4\,||A\mathbf{u}_0||^2\,, ||\sigma_0||_2\,, ||\tau_0||_2\}, \tag{22}$$

then $(\mathbf{u}_0, \sigma_0, \tau_0) \in \mathfrak{R}_\mathrm{T}$. In fact, $\mathbf{w}$ is a solution of problem

$$\left\{\begin{array}{rcl} \mathbf{w}(\cdot) \in \mathbf{H}^1(\Omega), & & \\ \mathbf{w}' + (1-\omega)A\mathbf{w} &=& 0, \quad \text{p.p. in } \mathbb{R}_+, \\ \mathbf{w}(0) &=& \mathbf{u}_0, \quad \text{in } \Omega, \\ \mathbf{w} &=& 0, \quad \text{on } \Sigma_\mathrm{T}. \end{array}\right. \tag{23}$$

Using estimate (17), there exists a constant C_4 such that

$$\|\mathbf{w}'\|^2_{\mathrm{L}^2(0,\mathrm{T},\mathbf{H}^3(\Omega))} + \|\mathbf{w}'\|^2_{\mathrm{L}^\infty(0,\mathrm{T},\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega))} + \|\mathbf{w}\|^2_{\mathrm{L}^2(0,\mathrm{T},\mathbf{H}^1_0(\Omega))} + \|\mathbf{w}\|^2_{\mathrm{L}^\infty(0,\mathrm{T},\mathbf{L}^2(\Omega))} \le C_4 \|A\mathbf{u}_0\|^2 .$$

Thus the choose of $\mathfrak{B}_1$ in (22) is enough for prove that $\mathfrak{R}_\mathrm{T}$ is non empty for each $\mathrm{T} > 0$.

Define now the application mapping $\mathfrak{K}$ in this way

$$\begin{array}{rccc} \mathfrak{K}: & \mathfrak{R}_\mathrm{T} & \longrightarrow & \mathfrak{X}_\mathrm{T} \\ & (\mathbf{w},\pi,\psi) & \longrightarrow & (\mathbf{u},\sigma,\tau) \end{array}$$

where $\mathfrak{X}_\mathrm{T} = \mathrm{C}([0,\mathrm{T}];\mathbf{H}^1_0(\Omega)) \times \mathrm{C}([0,\mathrm{T}];\mathrm{H}^1(\Omega)) \times \mathrm{C}([0,\mathrm{T}];\mathbf{H}^1(\Omega))$ and $\mathbf{u}$, σ and τ are solution of (15), (20) and (21), respectively, with

$$\begin{aligned} \mathfrak{F} &= \alpha\mathbf{f} + (1-\omega)\frac{\varepsilon^2\pi}{\alpha+\varepsilon^2\pi}A\mathbf{w} + \frac{\varepsilon^2}{\alpha+\varepsilon^2\pi}(\pi - w(\pi))\nabla\pi - \alpha(\mathbf{w}.\nabla)\mathbf{w} \\ &\quad -\nabla\pi + \mathbf{div}\,\psi, \\ \mathcal{G} &= -\varepsilon^{-2}\alpha\,\mathrm{div}\,\mathbf{w}. \end{aligned}$$

If we take

$$\begin{aligned} \mathfrak{B}_1 > \max\Big\{ & C_4\|A\mathbf{u}_0\|^2, e^{\sqrt{2}}\left(\|\sigma_0\|_2 + \|\tau_0\|_2 + 1 + \frac{2\omega}{C_3\mathrm{We}}\right), \\ & C_2(2C_5+1)\|A\mathbf{u}_0\|^2 + C_5\|A\mathbf{u}_0\|^4 + \\ & 3\Big(2(1+\|w\|^2_{\mathcal{C}})\|\sigma_0\|^2_1 + \|\tau_0\|^2_1\Big) + \\ & 3\|\mathbf{f}(0)\|^2 + 3\|\mathbf{f}\|^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{H}^1(\Omega))} + 3\|\mathbf{f}'\|^2_{\mathrm{L}^2(0,\mathrm{T},\mathrm{H}^{-1}(\Omega))}\Big\}, \end{aligned} \tag{24}$$

and

$$\mathfrak{B}_2 > e^{\sqrt{2}}\left\{C_6\left(\|\sigma_0\|_2 + \|\tau_0\|_2 + 1 + \frac{2\omega}{C_3\mathrm{We}}\right) + \frac{1}{\mathrm{We}}\left(\|\tau_0\|_2 + \frac{2\omega}{C_3\mathrm{We}}\right)\right\}, \tag{25}$$

and for all T small enough such that

$$\mathrm{T} \le \mathrm{T}^* = \min\left(\frac{\mathfrak{B}_1}{2C_2(4C_4(1+\|w\|^2_{\mathcal{C}})\mathfrak{B}_1^2 + 3\mathfrak{B}_2^2}, \frac{2}{C_6^2\mathfrak{B}_1}\right), \tag{26}$$

we have $\mathfrak{K}(\mathfrak{R}_\mathrm{T}) \subset \mathfrak{R}_\mathrm{T}$.

We now use Schauder fixed point theorem. The mapping $\mathfrak{K}$ is defined from convex, bounded and no empty set $\mathfrak{R}_\mathrm{T}$ into $\mathfrak{X}_\mathrm{T}$. To finish, we need to show the continuity of $\mathfrak{K}$ in $\mathfrak{X}_\mathrm{T}$.

Lemma 4.5 *To show the continuity of $\mathfrak{K}$ in $\mathfrak{X}_{\mathrm{T}}$, it is enough to show the continuity of $\mathfrak{K}$ in*

$$\mathfrak{Y}_{\mathrm{T}} = C([0,\mathrm{T}];\mathbf{L}^2(\Omega)) \times C([0,\mathrm{T}];\mathrm{L}^2(\Omega)) \times C([0,\mathrm{T}];\mathbf{L}^2(\Omega)).$$

Proof.
Let $\Big((\mathbf{w}_n,\pi_n,\psi_n)\Big)_n$ be a sequence of $\mathfrak{R}_{\mathrm{T}}$ and tends to $(\mathbf{w},\pi,\psi)$, such that:

$$(\mathbf{u}_n,\sigma_n,\tau_n) = \mathfrak{K}(\mathbf{w}_n,\pi_n,\psi_n) \text{ and } (\mathbf{u},\sigma,\tau) = \mathfrak{K}(\mathbf{w},\pi,\psi).$$

Suppose that $\mathfrak{K}$ is continuous in $\mathfrak{Y}_{\mathrm{T}}$, then the sequence $\Big(\mathfrak{K}(\mathbf{w}_n,\pi_n,\psi_n)_n\Big)$ tends to $\mathfrak{K}(\mathbf{w},\pi,\psi)$ in $\mathfrak{Y}_{\mathrm{T}}$, *i.e.*

$$\lim_{n\longrightarrow\infty} ||(\mathbf{u}_n,\sigma_n,\tau_n) - (\mathbf{u},\sigma,\tau)||_{\mathfrak{Y}_{\mathrm{T}}} = 0. \tag{27}$$

$\mathfrak{R}_{\mathrm{T}}$ is a compact set in $\mathfrak{X}_{\mathrm{T}}$ (see for instance [5]). Using (27), we can extract of $\Big((\mathbf{u}_n,\sigma_n,\tau_n)\Big)_n$ a subsequence converges in $\mathfrak{X}_{\mathrm{T}}$ to the unique accumulation point $(\mathbf{u},\sigma,\tau)$. Then the sequence $\Big((\mathbf{u}_n,\sigma_n,\tau_n)\Big)_n = \Big(\mathfrak{K}(\mathbf{w}_n,\pi_n,\psi_n)\Big)_n$ converges to $(\mathbf{u},\sigma,\tau) = \mathfrak{K}(\mathbf{w},\pi,\psi)$ in $\mathfrak{X}_{\mathrm{T}}$. This proved the continuity of $\mathfrak{K}$ in $\mathfrak{X}_{\mathrm{T}}$.
$\square$

Lemma 4.6 $\mathfrak{K}$ *is continuous in* $\mathfrak{Y}_{\mathrm{T}}$.

Proof.
Let $\Big((\mathbf{w}_n,\pi_n,\psi_n)\Big)_n$ be a sequence of $\mathfrak{R}_{\mathrm{T}}$ and tends to $(\mathbf{w},\pi,\psi)$, such that:

$$(\mathbf{u}_n,\sigma_n,\tau_n) = \mathfrak{K}(\mathbf{w}_n,\pi_n,\psi_n) \text{ and } (\mathbf{u},\sigma,\tau) = \mathfrak{K}(\mathbf{w},\pi,\psi).$$

Consider two systems. The first is :

$$\left\{\begin{array}{rcll} \alpha\mathbf{u}_n' + (1-\omega)A\mathbf{u}_n &=& \mathfrak{F}_n, & \\ \sigma_n' + (\mathbf{w}_n.\nabla)\sigma_n + \sigma_n \mathrm{div}\,\mathbf{w}_n &=& \mathcal{G}_n, & \\ \tau_n + \mathrm{We}\{\tau_n' + (\mathbf{w}_n.\nabla)\tau_n + \mathbf{g}(\nabla\mathbf{w}_n,\tau_n)\} &=& 2\omega\mathbf{D}[\mathbf{w}_n], & \text{in } \mathrm{Q}_{\mathrm{T}}, \\ \mathbf{u}_n(0,x) &=& \mathbf{u}_0(x), & \\ \sigma_n(0,x) &=& \sigma_0(x), & \\ \tau_n(0,x) &=& \tau_0(x), & \text{in } \Omega, \\ \mathbf{u}_n &=& 0, & \text{on } \Sigma_{\mathrm{T}}, \end{array}\right. \tag{28}$$

with

$$\begin{array}{rcl} \mathfrak{F}_n &=& \mathbf{F}(\mathbf{w}_n,\pi_n,\psi_n) - \alpha(\mathbf{w}_n.\nabla)\mathbf{w}_n - \nabla\pi_n + \mathbf{div}\,\psi_n, \\ \mathcal{G}_n &=& -\varepsilon^{-2}\mathrm{div}\,\mathbf{w}_n. \end{array}$$

And, the second is:

$$\left\{\begin{array}{rcll} \alpha\mathbf{u}' + (1-\omega)A\mathbf{u} &=& \mathfrak{F}, & \\ \sigma' + (\mathbf{w}.\nabla)\sigma + \sigma\mathrm{div}\,\mathbf{w} &=& \mathcal{G}, & \\ \tau + \mathrm{We}\{\tau' + (\mathbf{w}.\nabla)\tau + \mathbf{g}(\nabla\mathbf{w},\tau)\} &=& 2\omega\mathbf{D}[\mathbf{w}], & \text{in } \mathrm{Q_T}, \\ \mathbf{u}(0,x) &=& \mathbf{u}_0(x), & \\ \sigma(0,x) &=& \sigma_0(x), & \\ \tau(0,x) &=& \tau_0(x), & \text{in } \Omega, \\ \mathbf{u} &=& 0, & \text{on } \Sigma_\mathrm{T}, \end{array}\right. \tag{29}$$

with

$$\begin{array}{rcl} \mathfrak{F} &=& \mathbf{F}(\mathbf{w},\pi,\psi) - \alpha(\mathbf{w}.\nabla)\mathbf{w} - \nabla\pi + \mathbf{div}\,\psi, \\ \mathcal{G} &=& -\varepsilon^{-2}\mathrm{div}\,\mathbf{w}. \end{array}$$

Let $\mathbf{v}_n = \mathbf{u}_n - \mathbf{u}$, $q_n = \sigma_n - \sigma$ and $\mathbf{S}_n = \tau_n - \tau$. Using (28) and (29), we obtain, in $\mathrm{Q_T}$:

$$\left\{\begin{array}{rcl} \alpha\mathbf{v}_n' + (1-\omega)A\mathbf{v}_n &=& \mathfrak{F}_1, \\ q_n' + (\mathbf{w}_n.\nabla)q_n + q_n\mathrm{div}\,\mathbf{w}_n &=& \mathcal{G}_1, \\ \mathbf{S}_n + \mathrm{We}\{\mathbf{S}_n' + (\mathbf{w}_n.\nabla)\mathbf{S}_n + \mathbf{g}(\nabla\mathbf{w}_n,\mathbf{S}_n)\} &=& \mathcal{H}_1, \end{array}\right. \tag{30}$$

with the boundary conditions:

$$\left\{\begin{array}{rcll} \mathbf{v}_n(0,x) &=& 0, & \\ q_n(0,x) &=& 0, & \\ \mathbf{S}_n(0,x) &=& 0, & \text{in } \Omega, \\ \mathbf{v}_n &=& 0, & \text{on } \Sigma_\mathrm{T}, \end{array}\right. \tag{31}$$

such that:

$$\begin{array}{rcl} \mathfrak{F}_1 &=& \mathbf{F}(\mathbf{w}_n,\pi_n,\psi_n) - \mathbf{F}(\mathbf{w},\pi,\psi) - \alpha\Big[(\mathbf{w}_n.\nabla)\mathbf{w}_n - (\mathbf{w}.\nabla)\mathbf{w}\Big] \\ && -\nabla(\pi_n - \pi) + \mathbf{div}\,(\psi_n - \psi), \\ \mathcal{G}_1 &=& -\varepsilon^{-2}\mathrm{div}\,(\mathbf{w}_n - \mathbf{w}) - \big((\mathbf{w}_n - \mathbf{w}).\nabla\big)\sigma - \sigma\mathrm{div}\,(\mathbf{w}_n - \mathbf{w}), \\ \mathcal{H}_1 &=& 2\omega\mathbf{D}[\mathbf{w}_n - \mathbf{w}] - \mathrm{We}\Big\{\big((\mathbf{w}_n - \mathbf{w}).\nabla\big)\tau + \mathbf{g}\big(\nabla(\mathbf{w}_n - \mathbf{w}),\tau\big)\Big\}. \end{array}$$

First, multiply the equation $(30)_1$ by $\mathbf{v}_n$, and integrate over Ω. We get:

$$\begin{array}{l} \alpha\dfrac{\mathrm{d}}{\mathrm{dt}}\,||\mathbf{v}_n||^2 + (1-\omega)\Big(\,||\nabla\mathbf{v}_n||^2 + ||\mathrm{div}\,\mathbf{v}_n||^2\Big) \leq \\ \qquad ||\pi_n - \pi||_1^2 + ||\psi_n - \psi||_1^2 + 4\,||\mathbf{v}_n||^2 + \alpha^2\,||(\mathbf{w}_n.\nabla)\mathbf{w}_n - (\mathbf{w}.\nabla)\mathbf{w}||^2 \\ \qquad\qquad + ||\mathbf{F}(\mathbf{w}_n,\pi_n,\psi_n) - \mathbf{F}(\mathbf{w},\pi,\psi)||^2 . \end{array} \tag{32}$$

We now estimate the term $||\mathbf{F}(\mathbf{w}_n,\pi_n,\psi_n) - \mathbf{F}(\mathbf{w},\pi,\psi)||^2$ on the right hand of (32). Using the two inequalities :

$$\frac{\mathfrak{m}_1}{2} \leq \alpha + \varepsilon^2\pi(t,x) \leq 2\mathfrak{M}_1 \text{ and } \frac{\mathfrak{m}_1}{2} \leq \alpha + \varepsilon^2\pi_n(t,x) \leq 2\mathfrak{M}_1,$$

we obtain:

$$
\begin{aligned}
||\mathbf{F}(\mathbf{w}_n,\pi_n,\psi_n)-\mathbf{F}(\mathbf{w},\pi,\psi)||^2 \leq C_7\mathfrak{m}_1\varepsilon^4\Big[&(1-\omega)^2\,||\pi_n A\mathbf{w}_n-\pi A\mathbf{w}||^2\\
&+||\pi_n\nabla\pi_n-\pi\nabla\pi||^2+||w(\pi_n)\nabla\pi_n-w(\pi)\nabla\pi||^2\\
&+||\pi_n\mathbf{div}\,\mathbf{S}_n-\pi\mathbf{div}\,\mathbf{S}||^2\Big].
\end{aligned}
$$

Then, (32) satisfies:

$$
\alpha\frac{\mathrm{d}}{\mathrm{dt}}\,||\mathbf{v}_n||^2+(1-\omega)\Big(\,||\nabla\mathbf{v}_n||^2+||\mathrm{div}\,\mathbf{v}_n||^2\Big)\;\leq\; C_8\ell_n++4\,||\mathbf{v}_n||^2\,, \tag{33}
$$

with

$$
\begin{aligned}
\ell_n \;=\;& ||\pi_n-\pi||_1^2+||\psi_n-\psi||_1^2+\alpha\,||(\mathbf{w}_n.\nabla)\mathbf{w}_n-(\mathbf{w}.\nabla)\mathbf{w}||^2\\
&+(1-\omega)^2\,||\pi_n A\mathbf{w}_n-\pi A\mathbf{w}||^2+||\pi_n\nabla\pi_n-\pi\nabla\pi||^2\\
&+||w(\pi_n)\nabla\pi_n-w(\pi)\nabla\pi||^2+||\pi_n\mathbf{div}\,\mathbf{S}_n-\pi\mathbf{div}\,\mathbf{S}||^2\,.
\end{aligned}
$$

Second, multiply the equation $(30)_2$ by $\varepsilon^2 q_n$ and integrate over Ω. This yields:

$$
\begin{aligned}
\varepsilon^2\frac{\mathrm{d}}{\mathrm{dt}}\,||q_n||^2 \;&\leq\; (1+C_9\,||\sigma||_2)\,||\mathbf{w}_n-\mathbf{w}||_1^2+(1+C_{10}\,||\mathbf{w}_n||_3)\,||q_n||^2\\
&\leq\; (1+C_9\,||\sigma||_2)\,||\mathbf{w}_n-\mathbf{w}||_1^2+j_n\,||q_n||^2\,,
\end{aligned} \tag{34}
$$

with $j_n=1+C_{10}\,||\mathbf{w}_n||_3$.
Finally, multiply the equation $(30)_3$ by $\mathbf{S}_n/2\omega$, we obtain

$$
\begin{aligned}
\frac{\mathrm{We}}{2\omega}\frac{\mathrm{d}}{\mathrm{dt}}\,||\mathbf{S}_n||^2 \;&\leq\; \left(1+\frac{\mathrm{We}C_{11}}{2\omega}\,||\tau||_2\right)||\mathbf{w}_n-\mathbf{w}||_1^2\\
&\quad+\left(1+\frac{1}{2\omega}+\frac{\mathrm{We}C_{12}}{2\omega}\,||\mathbf{w}_n||_3\right)||\mathbf{S}_n||^2\\
&\leq\; \left(1+\frac{\mathrm{We}C_{11}}{2\omega}\,||\tau||_2\right)||\mathbf{w}_n-\mathbf{w}||_1^2+k_n\,||\mathbf{S}_n||^2\,,
\end{aligned} \tag{35}
$$

with $k_n=1+\frac{1}{2\omega}+\frac{\mathrm{We}C_{12}}{2\omega}\,||\mathbf{w}_n||_3$.

The functions j_n and k_n, are positive and, because of the class of solutions we consider, j_n and k_n belong to $L^1(0,T)$. Therefore, by using (33), (34), and (35), we deduce from Gronwall's lemma that:

$$
||\mathbf{v}_n||^2 \;\leq\; \frac{C_8}{\alpha}\int_0^t \exp\left(\frac{-4s}{\alpha}\right)\ell_n(s)ds, \tag{36}
$$

$$
||q_n||^2 \;\leq\; \frac{1}{\varepsilon^2}\left(1+C_9\mathfrak{B}_1\right)\int_s^t \exp\left(\int_0^s j_n(r)dr\right)||\mathbf{w}_n(s)-\mathbf{w}(s)||_1^2\,ds, \tag{37}
$$

$$
||\mathbf{S}_n||^2 \;\leq\; \left(\frac{2\omega}{\mathrm{We}}+C_{11}\mathfrak{B}_1\right)\int_s^t \exp\left(\int_0^s k_n(r)dr\right)||\mathbf{w}_n(s)-\mathbf{w}(s)||_1^2\,ds. \tag{38}
$$

The sequence $\Big((\mathbf{w}_n,\pi_n,\psi_n)\Big)_n$ of $\mathfrak{R}_{\mathrm{T}}$ tends to $(\mathbf{w},\pi,\psi)$, and using (36), (37) and (38), we obtain $\mathbf{v}_n$, q_n and $\mathbf{S}_n$ tend to zero in $\mathfrak{Y}_{\mathrm{T}}$. This meaning that the sequence $\Big((\mathbf{u}_n,\sigma_n,\tau_n)\Big)_n = \Big(\mathfrak{K}(\mathbf{w}_n,\pi_n,\psi_n)\Big)_n$ tends to $(\mathbf{u},\sigma,\tau) = \mathfrak{K}(\mathbf{w},\pi,\psi)$ and $\mathfrak{K}$ is continuous in $\mathfrak{Y}_{\mathrm{T}}$. □

4.4 Proof of Theorem 3.2

We take, as usual, the difference of two solutions $(\mathbf{u}_1,\sigma_1,\tau_1)$ and $(\mathbf{u}_2,\sigma_2,\tau_2)$ belonging to the class specified in the theorem 3.2. The vector function $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, the scalar function $\sigma = \sigma_1 - \sigma_2$ and the tensor function $\tau = \tau_1 - \tau_2$ satisfy the following system:

$$\left\{\begin{array}{lcl} \alpha\Big[\mathbf{u}' + (\mathbf{u}.\nabla)\mathbf{u}_1 + (\mathbf{u}_2.\nabla)\mathbf{u}\Big] + (1-\omega)A\mathbf{u} + \nabla\sigma - \mathbf{div}\,\tau & = & \mathfrak{F}_2, \\ \sigma' + (\mathbf{u}_1.\nabla)\sigma + (\mathbf{u}.\nabla)\sigma_2 + \sigma\mathrm{div}\,\mathbf{u}_1 + \sigma_2\mathrm{div}\,\mathbf{u} & = & -\varepsilon^{-2}\mathrm{div}\,\mathbf{u}, \\ \tau + \mathrm{We}\{\tau' + (\mathbf{u}_1.\nabla)\tau + \big(\mathbf{u}.\nabla\big)\tau_2 + \mathbf{g}(\nabla\mathbf{u}_1,\tau) + \mathbf{g}\big(\nabla\mathbf{u},\tau_2\big)\} & = & 2\omega\mathbf{D}[\mathbf{u}], \end{array}\right. \tag{39}$$

with the boundary conditions:

$$\left\{\begin{array}{rcll} \mathbf{u}(0,x) & = & 0, & \\ \sigma(0,x) & = & 0, & \\ \tau(0,x) & = & 0, & \text{in } \Omega, \\ \mathbf{u} & = & 0, & \text{on } \Sigma_{\mathrm{T}}, \end{array}\right. \tag{40}$$

such that:

$$\mathfrak{F}_2 \;=\; \mathbf{F}(\mathbf{u}_1,\sigma_1,\tau_1) - \mathbf{F}(\mathbf{u}_2,\sigma_2,\tau_2).$$

Multiply $(39)_1$, $(39)_2$ and $(39)_3$ by $\mathbf{u}$, $\varepsilon^2\sigma/\alpha$, and $\tau/(2\omega)$, respectively, and integrate over Ω. Summing the three obtained equations, one obtains

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\alpha||\mathbf{u}||^2 + \frac{\varepsilon^2}{\alpha}||\sigma||^2 + \frac{\mathrm{We}}{2\omega}||\tau||^2\right) + (1-\omega)\Big(||\nabla\mathbf{u}||^2 + ||\mathrm{div}\,\mathbf{u}||^2\Big) + \frac{1}{2\omega}\,||\tau||^2 \\ &\quad\le \alpha\mathrm{C}_{12}\Big[\,||\mathbf{u}_1||\,||\mathbf{u}||^2 + ||\mathbf{u}_2||\,||\mathbf{u}||^2\Big] \\ &\qquad + \frac{\varepsilon^2}{\alpha}\mathrm{C}_{12}\Big[\,||\sigma||\,||\mathbf{u}_1||_3\,||\nabla\mathbf{u}|| + ||\sigma_2||_2\,||\nabla\mathbf{u}||\,||\mathbf{u}|| \\ &\qquad + ||\sigma||\,||\sigma_1||_2\,||\mathrm{div}\,\mathbf{u}|| + ||\sigma_2||_2\,||\sigma||\,||\mathrm{div}\,\mathbf{u}|| \\ &\qquad + ||\tau_1||\,||\sigma||\,||\nabla\mathbf{u}|| + ||\sigma_2||\,||\tau||\,||\nabla\mathbf{u}||\,\Big] \\ &\qquad + \frac{\varepsilon^2}{\alpha}\mathrm{C}_{12}\Big[\,||\mathbf{u}_1||_3\,||\sigma||^2 + ||\nabla\mathbf{u}||\,||\sigma_2||_2\,||\sigma||\,\Big] \\ &\qquad + \frac{\mathrm{We}}{2\omega}\mathrm{C}_{12}\Big[\,||\mathbf{u}_1||_3\,||\tau||^2 + ||\nabla\mathbf{u}||\,||\tau_2||_2\,||\tau||\,\Big]. \end{aligned} \tag{41}$$

For $\delta > 0$, (41) can be written as:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\alpha||\mathbf{u}||^2 + \frac{\varepsilon^2}{\alpha}||\sigma||^2 + \frac{\mathrm{We}}{2\omega}||\tau||^2\right) + (1-\omega)\Big(||\nabla\mathbf{u}||^2 + ||\mathrm{div}\,\mathbf{u}||^2\Big) + \frac{1}{2\omega}\,||\tau||^2 \\
&\quad \leq \left[\mathrm{C}_{12}\big(||\mathbf{u}_1|| + ||\mathbf{u}_2||\big) + \frac{(\mathrm{C}_{12})^2}{2\delta}\,||\sigma_2||_2^2\right]\alpha\,||\mathbf{u}||^2 \\
&\quad + \frac{\delta}{2}\left[\left(\frac{5\varepsilon^2}{\alpha} + \frac{\mathrm{We}}{2\omega}\right)||\nabla\mathbf{u}||^2 + \frac{2\varepsilon^2}{\alpha}\,||\mathrm{div}\,\mathbf{u}||^2\right] \\
&\quad + \left[\frac{(\mathrm{C}_{12})^2}{2\delta}\Big(||\mathbf{u}_1||_2^3 + ||\sigma_1||_2^2 + 2\,||\sigma_2||_2^2 + ||\tau_1||_2^2\Big) + \mathrm{C}_{12}\,||\mathbf{u}_1||_3\right]\frac{\varepsilon^2}{\alpha}\,||\sigma||^2 \\
&\quad + \left[\frac{(\mathrm{C}_{12})^2}{2\delta}\,||\sigma_2||_2^2 + \mathrm{C}_{12}\,||\mathbf{u}_1||_3\right]\frac{\mathrm{We}}{2\omega}\,||\tau||^2\,. \qquad (42)
\end{aligned}$$

From (42), we then deduce that solutions $(\mathbf{u}, \sigma, \tau)$ of (39) satisfy the following energy inequality:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\alpha||\mathbf{u}||^2 + \frac{\varepsilon^2}{\alpha}||\sigma||^2 + \frac{\mathrm{We}}{2\omega}||\tau||^2\right) + (1-\omega)\Big(1 - \frac{\delta(10\varepsilon^2\omega + \alpha\mathrm{We})}{4\alpha\omega(1-\omega)}\Big)||\nabla\mathbf{u}||^2 \\
&\quad + (1-\omega)\Big(1 - \frac{\delta\varepsilon^2}{\alpha(1-\omega)}\Big)||\mathrm{div}\,\mathbf{u}||^2 + \frac{1}{2\omega}\,||\tau||^2 \leq \mathcal{X}_\delta\left[\alpha\,||\mathbf{u}||^2 + \frac{\varepsilon^2}{\alpha}||\sigma||^2\right. \\
&\quad \left. + \frac{\mathrm{We}}{2\omega}||\tau||^2\right], \qquad (43)
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{X}_\delta &= \mathrm{C}_{12}\big(||\mathbf{u}_1|| + ||\mathbf{u}_2|| + ||\mathbf{u}_1||_3\big) + \frac{(\mathrm{C}_{12})^2}{2\delta}\Big(||\mathbf{u}_1||_2^3 + ||\sigma_1||_2^2 \\
&\quad + 2\,||\sigma_2||_2^2 + ||\tau_1||_2^2\Big). \qquad (44)
\end{aligned}$$

The function $\mathcal{X}_\delta$, defined in (44), is positive. Moreover, because of the class of solutions we consider, $\mathcal{X}_\delta$ belongs to $L^1(0, T)$. Therefore, choosing $\delta > 0$ small enough, we deduce from Gronwall's lemma that $\mathbf{u} = 0$, $\sigma = 0$ and $\tau = 0$ in $\mathrm{Q_T}$, and that consequently $\mathbf{u}_1 = \mathbf{u}_2$, $\sigma_1 = \sigma_2$, $\tau_1 = \tau_2$ in $\mathrm{Q_T}$ and the system (39) has a unique solution.

5 Open Problem

A question can be posed about the possibility of adapting this result to improve the existence result in case of a bounded domain with a singularity (a convex angle by example). This latter, which is very important because of its real occurrence, needs a particular study of the solutions behavior in the convex corner.

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