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Inequalities in Fractional Integrals

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Abstract

In this paper, we use the Riemann-Liouville fractional integrals to generate some new integral inequalities involving concave functions. We also establish some other results using positive functions. Other fractional results using Qi inequalities are also presented.

Keywords: *Concave function, Riemann-Liouville integral, Qi inequality.*

1 Introduction

It is well known that the integral inequalities involving functions of real variables play a fundamental role in the theory of differential equations [2, 3, 8, 9]). Moreover, the fractional type inequalities are of great importance. We refer the reader to [1, 4, 5] for some applications.
In this paper we are concerned with some fractional integral inequalities using positive functions. We also use some concave functions to generate other fractional results. Also, we give some sufficient conditions to study other inequalities of Qi type [7].

2 Basic Definitions

Definition 2.1: A real valued function $f(t), t \geq 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where

$f_1(t) \in C([0,\infty[)$.

Definition 2.2: A function $f(t), t \geq 0$ is said to be in the space $C_\mu^n, \mu \in \mathbb{R}$, if $f^{(n)} \in C_\mu$.

Definition 2.3: The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu, (\mu \geq -1)$ is defined as

$$\begin{array}{c} J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0, \\ J^0 f(t) = f(t), \end{array} \tag{1}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

For the convenience of establishing the results, we give the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \alpha \geq 0, \beta \geq 0, \tag{2}$$

which implies the commutative property

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t). \tag{3}$$

For more details, one can consult [6].

3 Main Results

Theorem 3.1 *Let f and g be two positive continuous functions on $[0,\infty[$. Then for $t > 0, p > 1, q > 1,$ we have*

$$J^\alpha [fg(t)] \leq A J^\alpha [f^p(t)] + B J^\alpha [g^q(t)], \tag{4}$$

with

$$A = \frac{\|f\|_p^{1-p} \|g\|_q}{p}, B = \frac{\|f\|_p \|g\|_q^{1-q}}{q}.$$

Proof: Let us consider the quantities:

$F(\tau) = \frac{f(\tau)}{\|f\|_p}, G(\tau) = \frac{g(\tau)}{\|g\|_q}, \tau \in [0,t], t > 0.$

Using Young inequality [8], we can write

$$(FG)(\tau) \leq \frac{1}{p}(F(\tau))^p + \frac{1}{q}(G(\tau))^q, \tau \in [0,t]. \tag{5}$$

Multiplying both sides of (5) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$, then integrating the result with respect to τ from 0 to t, we obtain

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)\|f\|_p\|g\|_q}\int_0^t (t-\tau)^{\alpha-1}(fg)(\tau)d\tau \\ &\leq \frac{1}{p\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\frac{f(\tau)}{\|f\|_p}\right]^p d\tau + \frac{1}{q\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[\frac{f(\tau)}{\|g\|_q}\right]^q d\tau. \end{aligned} \tag{6}$$

Therefore,

$$\frac{J^\alpha[fg(t)]}{\|f\|_p\|g\|_q} \leq \frac{J^\alpha[f^p(t)]}{p\|f\|_p^p} + \frac{J^\alpha[g^q(t)]}{q\|g\|_p^q}. \tag{7}$$

Theorem 3.1 is thus proved.

Our second result is the following theorem.

Theorem 3.2 *Let f and g be two positive continuous and concave functions on $[0,\infty[$. Then for $t>0, \alpha>0$, we have*

$$\begin{aligned} &J^\alpha[fg(t)] \\ &\geq f(0)g(0)\frac{t^\alpha}{\Gamma(3+\alpha)} + [f(0)g(t)+f(t)g(0)]\frac{t^\alpha}{\Gamma(3+\alpha)} + (fg)(t)\frac{2t^{\alpha+1}}{\Gamma(\alpha+4)}. \end{aligned} \tag{8}$$

Proof: Since f and g are two concave functions, then for all $\tau \in [0,t]$, we can write

$$f(\tau) = f\left((1-\tfrac{\tau}{t}).0 + \tfrac{\tau}{t}.t\right) \geq \left((1-\tfrac{\tau}{t})f(0) + \tfrac{\tau}{t}f(t)\right) = h_1(\tau) \tag{9}$$

and

$$g(\tau) = g\left((1-\tfrac{\tau}{t}).0 + \tfrac{\tau}{t}.t\right) \geq \left((1-\tfrac{\tau}{t})g(0) + \tfrac{\tau}{t}g(t)\right) = h_2(\tau). \tag{10}$$

Therefore,

$$(fg)(\tau) \geq (h_1h_2)(\tau). \tag{11}$$

This implies that

$$\begin{aligned} &J^\alpha[fg(t)] \\ &\geq \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left[\left((1-\tfrac{\tau}{t})f(0)+\tfrac{\tau}{t}f(t)\right)\left((1-\tfrac{\tau}{t})g(0)+\tfrac{\tau}{t}g(t)\right)\right]d\tau \\ &\geq \frac{1}{t^2}\left[f(0)g(0)J^{\alpha+2}(1) + (f(0)g(t)+g(0)f(t))\,J^{\alpha+1}(t) + f(t)g(t)J^{\alpha+1}(t^2)\right]. \end{aligned} \tag{12}$$

With simple calculations, we obtain (4).
The following result has some relationship with the paper [7].

Theorem 3.3 *Suppose that* $\alpha > 0, \delta > 0, 1 < \beta \leq 2, \gamma \geq 2\alpha + 1$ *and let* $f \in C^1[0, \infty[$ *such as* $f(0) = 0 \leq f'(x) \leq M; x \in [0, t]$ *and*

$$\beta(\beta - 1)(t - x)^{\delta-1} x^{\beta-1} (Mx)^{\alpha\beta-\gamma} \delta^{2-\beta} - (\gamma - \alpha). \tag{13}$$

Then the inequality

$$\left[J^\delta[f^\alpha(t)]\right]^\beta \geq \Gamma^{-\beta+1}(\delta) J^\delta \left[f^\gamma(t)\right] \tag{14}$$

is valid.

Proof: Let us take $t > 0$. From $f(0) = 0, 0 \leq f'(\tau) \leq M, \tau \in [0, x], x \leq t$, we obtain

$$0 \leq f^\alpha(\tau) \leq (Mx)^\alpha. \tag{15}$$

Multiplying both sides of (15) by $(t - \tau)^{\delta-1}$ and integrating the result over $[0, x]$ with $x \leq t$, we obtain

$$0 \leq \int_0^x (t - \tau)^{\delta-1} f^\alpha(\tau) d\tau \leq \int_0^x (t - \tau)^{\delta-1} (Mx)^\alpha d\tau,$$

Hence for all $0 \leq x \leq t$, we have

$$0 \leq \int_0^x (t - \tau)^{\delta-1} f^\alpha(\tau) d\tau \leq \frac{t^\delta}{\delta} (Mx)^\alpha. \tag{16}$$

Now, we define the function

$$F(x) = \left(\int_0^x (t - \tau)^{\delta-1} f^\alpha(\tau) d\tau\right)^\beta - \int_0^x (t - \tau)^{\delta-1} f^\gamma(\tau) d\tau.$$

It's clear that $F(0) = 0$ and

$$\begin{aligned} F'(x) &= \beta(t - x)^{\delta-1} f^\alpha(x) \left(\int_0^x (t - \tau)^{\delta-1} f^\alpha(\tau) d\tau\right)^{\beta-1} - (t - x)^{\delta-1} f^\gamma(x) \\ &= (t - x)^{\delta-1} f^\alpha(x) \Big(\beta(\int_0^x (t - \tau)^{\delta-1} f^\alpha(\tau) d\tau)^{\beta-1} - f^{\gamma-\alpha}(x)\Big). \end{aligned} \tag{17}$$

We notice that $F'(x) = (t - x)^{\delta-1} f^\alpha(x) G(x)$, with

$$G(x) = \beta \left(\int_0^x (t - \tau)^{\delta-1} f^\alpha(\tau) d\tau\right)^{\beta-1} - f^{\gamma-\alpha}(x).$$

We have $G(0) = 0$ and

$$\begin{aligned} G'(x) = \beta(\beta - 1)(t - x)^{\delta-1} f^\alpha(x) \Big(\int_0^x (t - \tau)^{\delta-1} f^\alpha(\tau) d\tau\Big)^{\beta-2} \\ -(\gamma - \alpha) f'(x) f^{\gamma-\alpha-1}(x). \end{aligned} \tag{18}$$

We set $G'(x) = f^{\alpha}(x)H(x)$, where

$$H(x) = \beta(\beta-1)(t-x)^{\delta-1}\left(\int_0^x (t-\tau)^{\delta-1} f^{\alpha}(\tau)d\tau\right)^{\beta-2} - (\gamma-\alpha)f'(x)f^{\gamma-2\alpha-1}(x).$$

From the conditions of Theorem 3.3 and the inequality (15), we get

$$H(x) \geq \beta(\beta-1)(t-x)^{\delta-1}\left(M^{\alpha}\frac{x^{\alpha+\delta}}{\delta}\right)^{\beta-2} - (\gamma-\alpha)M(Mx)^{\gamma-2\alpha-1}. \quad (19)$$

Therefore,

$$H(x) \geq M^{\gamma-2\alpha}x^{(\alpha+1)(\beta-2)}\left[\beta(\beta-1)(t-x)^{\delta-1}M^{\alpha\beta-\gamma}\delta^{2-\beta} - (\gamma-\alpha)x^{\gamma-\alpha\beta-\beta+1}\right]. \quad (20)$$

Thanks to (13), we deduce that $H(x) \geq 0$, which implies that $G'(x) \geq 0$.
Now, since $G(0) = 0$, we deduce that $G(x) \geq 0, x \in [0, t]$.
Finally $F'(x) \geq 0$. And since $F(0) = 0$, then $F(x) \geq 0$. Hence, we can write

$$\left(\int_0^x (t-\tau)^{\delta-1} f^{\alpha}(\tau)d\tau\right)^{\beta} \geq \int_0^x (t-\tau)^{\delta-1} f^{\gamma}(\tau)d\tau.$$

Therefore,

$$\left[J^{\delta}[f^{\alpha}(t)]\right]^{\beta} \geq \Gamma^{-\beta+1}(\delta)J^{\delta}\left[f^{\gamma}(t)\right].$$

In particular, for $x = t$, we get the desired inequality (14).

4 Open Problems

In this paper, we have investigated some inequalities of Qi type for fractional integral based on [7] and other inequalities for cancave and positive functions. We will continue exploring other inequalities of this type. At the end, we pose the following problems:
Open Problem 1. In Theorem 3.1, can we find the best constants for the inequality (4)?
Open Problem 2. Do there exist other weak conditions for the results in Theorem 3.3?

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