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On a Mean Interpolating the Logarithmic and Identric Means

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Abstract

In this paper, we give a positive answer for an open problem posed by Raïssouli about a new mean defined in terms of the parameterized logarithmic mean, [1].

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1 Introduction

The following

$$L(a,b) = \frac{b-a}{\log b - \log a}, \quad a,b>0, \quad a \neq b, \quad L(a,a) = a \tag{1}$$

and,

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a,b>0,\ a\neq b, \quad I(a,a)=a \tag{2}$$

are known in the literature as the Logarithmic mean and Identric mean, respectively. In [1], the next mean has been introduced

$$E(a,b) = \int_0^1 L_t(a,b)dt, \quad a,b>0, \tag{3}$$

where

$$L_t(a,b) = \prod_{n=1}^{\infty} \left((1-t)a^{\frac{1}{2^n}} + tb^{\frac{1}{2^n}} \right), \quad a,b > 0, \quad a \neq b, \quad t \in [0,1], \qquad (4)$$

stands for the parameterized logarithmic mean [1].

We recall that a parameterized mean $m_\alpha(a,b)$, $\alpha \in [0,1]$, is a binary map between positive real numbers a, b satisfying:

- $m_\alpha(a,a) = a \quad \text{for} \quad a > 0,\ \alpha \in [0,1];$

- $m_\alpha(ta,tb) = tm_\alpha(a,b) \quad \text{for} \quad a,b,t > 0,\ \alpha \in [0,1];$

- $m_\alpha(a,b) \quad \text{is increasing in } a \text{ (and in } b), \text{ for each } \alpha \in [0,1];$

- $m_\alpha(a,b) = m_{1-\alpha}(b,a) \quad \text{for} \quad \alpha \in [0,1],\ a,b > 0;$

- $m_{\frac{1}{2}}(a,b) = m(a,b) \quad \text{for} \quad a,b > 0 \quad \text{where} \quad m(a,b) \quad \text{is a mean.}$

For some other details about parameterized means, see [1] and the related references cited there in.

The next open problem has been stated by M. Raïssouli in [1].

Open Problem. Prove or disprove that the means E and I are different. We conjecture that E interpolates L and I, i. e. $L < E < I$,

As already pointed before, our aim in this paper is to give a positive answer for the above problem.

2 Main results

The proof of our result will be based on the Hermite-Hadamard Inequality and Lemma 2.2 which we will state below.
First, we recall the Hermite-Hadamard Inequality [2]

Theorem 2.1 *If $f : [a,b] \to \mathbb{R}$ is a convex function, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

If f is strictly convex then the above inequalities are strict.

Second, we prove the next lemma.

Lemma 2.2 *Let* $g_s(t) = \prod_{n=1}^{\infty}\left((1-t)+ts^{\frac{1}{2^n}}\right)$ *for* $t \in [0,1]$, $s \in (0,1)$. *Then* $g_s(t)$ *is a strict convex function on* $[0,1]$.

Proof. From

$$g_s(t) = e^{\sum\limits_{n=1}^{\infty}\ln\left(1-t+ts^{\frac{1}{2^n}}\right)}$$

we have

$$g_s'(t) = g_s(t)\sum_{n=1}^{\infty}\frac{s^{\frac{1}{2^n}}-1}{1-t+ts^{\frac{1}{2^n}}},$$

$$g_s''(t) = g_s(t)\left(\left(\sum_{n=1}^{\infty}\frac{s^{\frac{1}{2^n}}-1}{1-t+ts^{\frac{1}{2^n}}}\right)^2 - \sum_{n=1}^{\infty}\frac{\left(s^{\frac{1}{2^n}}-1\right)^2}{\left(1-t+ts^{\frac{1}{2^n}}\right)^2}\right).$$

Because $g_s''(t) > 0$, $t \in [0,1]$ we get $g_s(t)$ is a strict convex function in the argument t for fixed s.

Now we are in position to state our main result.

Theorem 2.3 *Let* $a, b > 0$, $a \neq b$. *With the above, the next double inequality holds*

$$L(a,b) < E(a,b) < I(a,b). \tag{5}$$

Proof. We note that the inequality $L(a,b) < E(a,b)$ was proved in [1] (see Proposition 3.1, p. 867). By using Hermite-Hadamard Inequality (Theorem 2.1) a more simple proof can be obtained. Really, we have

$$\int_0^1 g_s(t)dt > g_s\left(\frac{1}{2}\right) = L(1,s).$$

By the homogeneity of E and L the inequality $L(a,b) < E(a,b)$ follows.

Now we show the inequality $E(a,b) < I(a,b)$. The right inequality in (5) can be rewritten as

$$\int_0^1 \prod_{n=1}^{\infty}\left((1-t)a^{\frac{1}{2^n}} + tb^{\frac{1}{2^n}}\right)dt < \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}. \tag{6}$$

Since E and I are homogeneous means we can, without loss the generality, assume that $0 < b < a$. Let us set $s = b/a$, then $0 < s < 1$. Inequality (6) is then reduced to

$$\int_0^1 \prod_{n=1}^{\infty} \left(1 - t + ts^{\frac{1}{2^n}}\right) dt = \int_0^1 g_s(t)dt < e^{\frac{s\ln s}{s-1}-1}.$$

Denote

$$F(s) = e^{\frac{s\ln s}{s-1}-1} - \int_0^1 \prod_{n=1}^{\infty} \left(1 - t + ts^{\frac{1}{2^n}}\right) dt.$$

If we show $F(s) > 0$ for $0 < s < 1$, then the proof will be complete. Lemma 2.2 and Hermite-Hadamard Inequality (Theorem 2.1.) imply

$$\int_0^1 g_s(t)dt = \int_0^{\frac{1}{2}} g_s(t)dt + \int_{\frac{1}{2}}^1 g_s(t)dt < \frac{1}{2}\left(\frac{1+\frac{s-1}{\ln s}}{2}\right) + \frac{1}{2}\left(\frac{\frac{s-1}{\ln s}+s}{2}\right) =$$

$$\frac{1+s}{4} + \frac{s-1}{2\ln s},$$

where we used $g_s(1/2) = (s-1)/\ln s$ and $L(a,b) = \prod_{n=1}^{\infty} \frac{a^{1/2^n}+b^{1/2^n}}{2}$, see [1].

Because

$$F(s) > e^{\frac{s\ln s}{s-1}-1} - \left(\frac{1+s}{4} + \frac{s-1}{2\ln s}\right),$$

it suffices to show that

$$\varphi(s) = \frac{s\ln s}{s-1} - 1 - \ln\left(\frac{1+s}{4} + \frac{s-1}{2\ln s}\right) > 0 \quad \text{for} \quad 0 < s < 1.$$

Simple computation gives

$$\varphi(1) = 0 \quad \text{and} \quad \varphi'(s) = \frac{s - \ln s - 1}{(s-1)^2} - \frac{s\ln^2 s + 2s\ln s - 2s + 2}{s\ln s(\ln s + s\ln s + 2s - 2)}$$

for $0 < s < 1$.

If we show that $\varphi'(s) < 0$, the proof will be complete.
Denote $v(s) = \ln s + s\ln s + 2s - 2$. From $v(1) = 0$, $v'(s) = \frac{1}{s} + \ln s + 3$, $v'(1) = 4$, $v''(s) = \frac{-1+s}{s^2} < 0$ we have $v(s) < 0$. It implies $\varphi'(s) < 0$ is equivalent to

$$\varphi\varphi(s) = \ln^3 s - \frac{2(s-1)^3}{s(1+s)} > 0$$

and to

$$\varphi\varphi\varphi(s) = \ln s + \frac{\sqrt[3]{2}(1-s)}{\sqrt[3]{s(1+s)}} > 0.$$

Because $\varphi\varphi\varphi(1) = 0$ it suffices to show that $\varphi\varphi\varphi'(s) < 0$. Simple computation gives

$$\varphi\varphi\varphi'(s) = \frac{1}{s} + \sqrt[3]{2}\left(-\frac{1}{\sqrt[3]{s(+s)}} - \frac{1-s}{3}\left(\frac{1}{s\sqrt[3]{s(1+s)}} + \frac{1}{(1+s)\sqrt[3]{s(1+s)}}\right)\right).$$

$\varphi\varphi\varphi'(s) < 0$ is equivalent to

$$(1+s)\sqrt[3]{s(1+s)} < \sqrt[3]{2}\left(s(1+s) + \frac{(1-s)(1+2s)}{3}\right).$$

It can be rewriting as

$$27s(1+s)^4 < 2(s^2+4s+1)^3.$$

Some computation gives that this inequality is

$$2(1+s)^6 - 15s(1+s)^4 + 24s^2(1+s)^2 + 16s^3 > 0.$$

This can be rewriting as

$$vv(s) = 2(1+s)^6 - 15(1+s)^5 + 39(1+s)^4 - 32(1+s)^3 - 24(1+s)^2 + 48(1+s) - 16 > 0$$

for $s \in (0,1)$.
But this is evident because of

$$vv(s) = 2(u-2)^4(u+1)\left(u - \frac{1}{2}\right) = 2(s-1)^4(s+2)\left(s + \frac{1}{2}\right), \quad u = 1+s.$$

The proof is complete.

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