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Some results on semiprime rings with generalized derivations

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Abstract

Let R be a semiprime ring, I a nonzero ideal of R and F is a generalized derivation associted with a derivation d. If F satisfies any one of following conditions: (i) $F(xy) - xy \in Z(R)$, (ii) $F(xy) - yx \in Z(R)$, (iii) $F(x)F(y) - xy \in Z(R)$, (iv) $F(x)F(y) - yx \in Z(R)$ for all $x, y \in I$, then $[I, R]d(R) = (0)$. In particular if R is prime, then either R is commutative or $d = 0$.

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1 Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. Recall that R is called prime if $aRb = (0)$ implies $a = 0$ or $b = 0$; it is semiprime if $aRa = (0)$ implies that $a = 0$. Clearly, every prime ring is a semiprime ring. For $x, y \in R$, $[x, y] = xy - yx$ (resp. $x \circ y = xy + yx$) denote the commutator (resp. the anticommutator) of x, y. An additive mapping $d : R \longrightarrow R$ is a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is a generalized derivation if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.
Several authors have proved commutativity theorems for prime and semiprime rings admitting derivations or generalized derivations satisfying any one of the properties $(i) - (iv)$ on any appropriate subset. Motivated by this result, our aim in the following paper is to study generalized derivation satisfying properties $(i) - (iv)$ on a a nonzero ideal of a semiprime ring.

2 Main results

In order to prove our theorems we shall need the following facts.

Fact 2.1 *Let R be a semiprime ring and I a nonzero ideal of R. If $x \in I$ is such that $xIx = (0)$, then $x = 0$. In particular, if $xI = (0)$, then $x = 0$.*

Fact 2.2 *Let R be a semiprime ring and I a nonzero ideal of R. If $xy \in Z(R)$ for all $x, y \in I$, then $I \subseteq Z(R)$.*

Proof. We have $[x, r]y + x[y, r] = 0$, for all $x, y \in I$, $r \in R$, replacing y by yx we get $xy[x, r] = 0$, left multiplying last expression by r we obtain $rxy[x, r] = 0$ again replacing y by ry implies that $xry[x, r] = 0$ so $[x, r]y[x, r] = 0$, that is $[x, r]I[x, r] = 0$, thus the semiprimeness together with Fact 2.1 yield $[x, r] = 0$ so that $I \subseteq Z(R)$. ∎

Theorem 2.3 *Let R be a semiprime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d satisfying $F(xy) - xy \in Z(R)$ for all $x, y \in I$, then $[I, R]d(R) = 0$. In particular if R is prime, then either R is commutative or $d = 0$.*

Proof. We are given that

$$F(xy) - xy \in Z(R) \quad \text{for all } x, y \in I. \tag{1}$$

Replacing y by yz in (1), where $z \in I$, we get $F(xy)z - xyz + xyd(z) \in Z(R)$ that is $[xyd(z), z] = 0$, so that

$$xy[d(z), z] + x[y, z]d(z) + [x, z]yd(z) = 0 \quad \text{for all } x, y, z \in I. \tag{2}$$

Writing $d(z)x$ instead of x in (2) we obtain

$$[d(z), z]xyd(z) = 0 \quad \text{for all } x, y, z \in I. \tag{3}$$

Substituting yz for y in (3) we get

$$[d(z), z]xyzd(z) = 0 \quad \text{for all } x, y, z \in I. \tag{4}$$

Right multiplying equation (3) by z we find that

$$[d(z), z]xyd(z)z = 0 \quad \text{for all } x, y, z \in I. \tag{5}$$

Employing equations (4) and (5) yield

$$[d(z), z]xy[d(z), z] = 0 \quad \text{for all } x, y, z \in I. \tag{6}$$

Right multiplying equation (6) by x leads to

$$[d(z), z]xI[d(z), z]x = 0 \quad \text{for all} \;\; x, z \in I. \tag{7}$$

Hence Fact 2.1 forces that

$$[d(z), z]x = 0 \quad \text{for all} \;\; x, z \in I, \tag{8}$$

once again applying Fact 2.1 we get

$$[d(z), z] = 0 \quad \text{for all} \;\; z \in I. \tag{9}$$

In view of ([7], main Theorem), equation (9) yields $[I, R]d(R) = 0$. ∎

Theorem 2.4 *Let R be a semiprime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d satisfying $F(xy) - yx \in Z(R)$ for all $x, y \in I$, then $[I, R]d(R) = 0$. In particular if R is prime, then either R is commutative or $d = 0$.*

Proof. We have

$$F(xy) - yx \in Z(R) \quad \text{for all} \;\; x, y \in I. \tag{10}$$

Replacing y by yz in (10), where $z \in R$, we get

$$[y, z][x, z]+y[[x, z], z]+[x, z]yd(z)+x[y, z]d(z)+xy[d(z), z] = 0 \quad \text{for all} \;\; x, y \in I, \;\; z \in R. \tag{11}$$

Substituting xy for y implies that

$$[x, z]y[x, z] + [x, z]xyd(z) = 0 \quad \text{for all} \;\; x, y \in I, \;\; z \in R. \tag{12}$$

Replacing z by $z + x$ we obtain

$$[x, z]xyd(x) = 0 \quad \text{for all} \;\; x, y \in I, \;\; z \in R. \tag{13}$$

Writing $d(x)$ instead of z in (13) we get

$$[d(x), x]xyd(x) = 0 \quad \text{for all} \;\; x, y \in I. \tag{14}$$

Replacing y by yx in (14) we get

$$[d(x), x]xyxd(x) = 0 \quad \text{for all} \;\; x, y \in I. \tag{15}$$

Right multiplying equation (14) by x we obtain

$$[d(x), x]xyd(x)x = 0 \quad \text{for all} \;\; x, y \in I. \tag{16}$$

Employing equations (15) and (16) we arrive at $[d(x),x]xy[d(x),x] = 0$, so that

$$[d(x),x]xI[d(x),x]x = 0 \quad \text{for all } \ x \in I. \tag{17}$$

In view of equation (17) Fact 2.1 forces that $[d(x),x]x = 0$, so that

$$[d(x),x]x^2 = 0 \quad \text{for all } \ x \in I. \tag{18}$$

On the other hand replacing y by x^2 in (10) we get

$$x^2[d(x),x] = 0 \quad \text{for all } \ x \in I. \tag{19}$$

Comparing equations (18) and (19) we conclude that $[[d(x),x],x^2] = 0$, so

$$[[d(x^2),x^2],x^2] = 0 \quad \text{for all } \ x \in I. \tag{20}$$

Accordingly, ([7], main Theorem) assures that $[I,R]d(R) = 0$. ∎

Theorem 2.5 *Let R be a semiprime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d satisfying $F(x)F(y) - xy \in Z(R)$ for all $x,y \in I$, then $[I,R]d(R) = 0$. In particular if R is prime, then either R is commutative or $d = 0$.*

Proof. Suppose that

$$F(x)F(y) - xy \in Z(R) \quad \text{for all } \ x,y \in I. \tag{21}$$

Replacing y by yz, where $z \in I$, we get

$$[F(x)yd(z),z] = 0 \quad \text{for all } \ x,y,z \in I. \tag{22}$$

Writing zy instead of y in (22) we obtain

$$[F(x)zyd(z),z] = 0 \quad \text{for all } \ x,y,z \in I. \tag{23}$$

Substituting xz for x in (22) and using (23) leads to

$$[xd(z)yd(z),z] = 0 \quad \text{for all } \ x,y,z \in I. \tag{24}$$

Replacing x by xz we arrive at

$$[xzd(z)yd(z),z] = 0 \quad \text{for all } \ x,y,z \in I. \tag{25}$$

Right multiplying (24) by z implies that

$$[xd(z)yd(z)z,z] = 0 \quad \text{for all } \ x,y,z \in I. \tag{26}$$

Subtracting (26) from (25) we find that

$$[x[d(z)yd(z),z],z]=0 \quad \text{for all} \;\; x,y,z\in I. \tag{27}$$

Putting $d(z)yd(z)x$ instead of x in (27) we obtain

$$[d(z)yd(z),z]I[d(z)yd(z),z]=0 \quad \text{for all} \;\; y,z\in I. \tag{28}$$

Applying Fact 2.1 we deduce

$$[d(z)yd(z),z]=0 \quad \text{for all} \;\; y,z\in I \tag{29}$$

Now replacing y by $yd(z)x$ where $x\in I$ we obtain

$$d(z)y[d(z),z]xd(z)=0 \quad \text{for all} \;\; x,y,z\in I. \tag{30}$$

Substituting zy for y yield

$$d(z)zy[d(z),z]xd(z)=0 \quad \text{for all} \;\; x,y,z\in I. \tag{31}$$

Left multiplying equation (30) by z we get

$$zd(z)y[d(z),z]xd(z)=0 \quad \text{for all} \;\; x,y,z\in I. \tag{32}$$

Subtracting (32) from (31) we arrive at

$$[d(z),z]y[d(z),z]xd(z)=0 \quad \text{for all} \;\; x,y,z\in I. \tag{33}$$

Replacing y by $xd(z)y$ implies that

$$[d(z),z]xd(z)I[d(z),z]xd(z)=0 \quad \text{for all} \;\; x,z\in I. \tag{34}$$

Hence

$$[d(z),z]xd(z)=0 \quad \text{for all} \;\; x,z\in I. \tag{35}$$

Replacing x by xz in (35) we get

$$[d(z),z]xzd(z)=0 \quad \text{for all} \;\; x,z\in I. \tag{36}$$

Right multiplying (35) by z we result

$$[d(z),z]xd(z)z=0 \quad \text{for all} \;\; x,z\in I. \tag{37}$$

Thus

$$[d(z),z]I[d(z),z]=0 \quad \text{for all} \;\; z\in I, \tag{38}$$

so that

$$[d(z),z]=0 \quad \text{for all} \;\; z\in I. \tag{39}$$

Applying ([7], main Theorem) we conclude that $[I,R]d(R)=0$. ∎

Theorem 2.6 *Let R be a semiprime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d satisfying $F(x)F(y) - yx \in Z(R)$ for all $x, y \in I$, then $[I, R]d(R) = 0$. In particular if R is prime, then either R is commutative or $d = 0$.*

Proof. We have

$$F(x)F(y) - yx \in Z(R) \quad \text{for all} \ \ x, y \in I. \tag{40}$$

Replacing y by yz, where $z \in I$, we get

$$(F(x)F(y) - yx)z + y[x, z] + F(x)yd(z) \in Z(R) \quad \text{for all} \ \ x, y, z \in I, \tag{41}$$

so that

$$[y[x, z], z] + [F(x)yd(z), z] = 0 \quad \text{for all} \ \ x, y, z \in I. \tag{42}$$

Writing xz instead of x, we obtain

$$[y[x, z], z]z + [F(x)zyd(z) + xd(z)yd(z), z] = 0 \quad \text{for all} \ \ x, y, z \in I. \tag{43}$$

Again replacing y by zy in (42) we arrive at

$$z[y[x, z], z] + [F(x)zyd(z), z] = 0 \quad \text{for all} \ \ x, y, z \in I. \tag{44}$$

Subtracting (44) from (43) we get

$$[[y[x, z], z], z] + [xd(z)yd(z), z] = 0 \quad \text{for all} \ \ x, y, z \in I. \tag{45}$$

Substituting xz for x in (45) we obtain

$$[[y[x, z], z], z]z + [xzd(z)yd(z), z] = 0 \quad \text{for all} \ \ x, y, z \in I, \tag{46}$$

Right multiplying equation (45) by z lead to

$$[[y[x, z], z], z]z + [xd(z)yd(z)z, z] = 0 \quad \text{for all} \ \ x, y, z \in I, \tag{47}$$

Employing (46) and (47) yield

$$[x[d(z)yd(z), z], z] = 0 \quad \text{for all} \ \ x, y, z \in I, \tag{48}$$

Since (48) is the same as (27) reasoning as in Theorem 2.5, we get the required result. ∎

3 Open Problems

To end this paper we introduce the following open questions :

(i) Does the condition $F([x, y]) - [x, y] \in Z(R)$ for all $x, y \in I$ implies that $[I, R]d(R) = 0$?

(ii) Does the condition $F(x \circ y) - x \circ y \in Z(R)$ for all $x, y \in I$ implies that $[I, R]d(R) = 0$?

(iii) Does the condition $[F(x), F(y)] - [x, y] \in Z(R)$ for all $x, y \in I$ implies that $[I, R]d(R) = 0$?

(iv) Does the condition $F(x) \circ F(y) - x \circ y \in Z(R)$ for all $x, y \in I$ implies that $[I, R]d(R) = 0$?

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