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New Fractional Results For a Boundary Value Problem With Caputo Derivative

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Abstract

In this paper, we consider three point boundary value problem for fractional differential equations of order $1 < \alpha < 2$. We establish new conditions for the existence and uniqueness of solutions by using Banach fixed point theorem. We also generate other existence results using Scheafer and Krasnoselskii fixed point theorems.

Keywords: *Boundary value problem, Caputo derivative, Fixed point theorem.*
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1 Introduction

The theory of differential equations of fractional order arises in many scientific disciplines, such as physics, chemistry, electrochemistry, control theory, image and signal processing, biophysics. For more details, we refer the reader to [3, 5, 9, 10, 12, 13, 14, 15, 17] and references therein. There has been a significant progress in the investigation of these equations in recent years, (see [4, 6, 7, 12, 16]). More recently, some basic theory for the initial boundary value problems of fractional differential equations has been discussed in [1, 2, 11, 12]. Motivated by the classical problem (1.1)-(1.2) in [8], this paper deals with the existence of solution for three point boundary value problems for the following problem

$$\begin{array}{l} D^{\alpha}x\left(t\right)+f\left(t,x\left(t\right)\right)=0, t\in J, 1<\alpha<2, \\ x\left(0\right)-\beta_{1}x^{'}\left(0\right)=0, x\left(1\right)-\beta_{2}x^{'}\left(\eta\right)=0, 0<\eta<1, \end{array} \tag{1}$$

where D^{α} denote the fractional derivative of order α in the sense of Caputo, $J=[0,1], \beta_1, \beta_2$ are real constants with $\beta_1+1\neq\beta_2$, f is a continuous function on $J\times\mathbb{R}$.

2 Preliminaries

In the following, we give the necessary notation and basic definitions which will be used in this paper.

Definition 2.1: The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$, for a continuous function f on $[0,\infty[$ is defined as

$$\begin{array}{c} J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau;\quad \alpha>0, t>0, \\ J^0f(t)=f(t), \end{array} \tag{2}$$

where $\Gamma(\alpha):=\int_0^{\infty}e^{-u}u^{\alpha-1}du.$

Definition 2.2: The fractional derivative of $f\in C^n([0,\infty[)$ in the Caputo's sense is defined as

$$D^{\alpha}f(t)=\left\{\begin{array}{ll} \frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau, & n-1<\alpha<n, n\in N^*, \\ \frac{d^n}{dt^n}f(t), & \alpha=n. \end{array}\right. \tag{3}$$

Details on Caputo's derivative can be found in [12, 15].
Let us now introduce $C\left(J,\mathbb{R}\right)$ the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm $\parallel x\parallel=\sup_{t\in J}\left|x\left(t\right)\right|$.
We give the following lemmas [11]:

Lemma 2.1 *For $\alpha>0$, the general solution of the fractional differential $D^{\alpha}x=0$ is given by*

$$x(t)=c_0+c_1t+c_2t^2+...c_{n-1}t^{n-1}, \tag{4}$$

where $c_i\in\mathbb{R}, i=0,1,2,...n-1, n=[\alpha]+1$.

Lemma 2.2 *Let $\alpha>0$, then*

$$J^{\alpha}D^{\alpha}x(t)=x(t)+c_0+c_1t+c_2t^2+...c_{n-1}t^{n-1}, \tag{5}$$

for some $c_i\in\mathbb{R}, i=0,1,2,...n-1, n=[\alpha]+1$.

We need also the following lemma:

Lemma 2.3 *Let $1<\alpha<2$. A solution of (1) is given by:*

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,x(\tau))\,d\tau + \frac{\beta_1+t}{(\beta_1+1-\beta_2)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau,x(\tau))\,d\tau \\ &\quad -\frac{\beta_2(\beta_1+t)}{(\beta_1+1-\beta_2)\Gamma(\alpha-1)}\int_0^\eta (\eta-\tau)^{\alpha-2} f(\tau,x(\tau))\,d\tau. \end{aligned} \tag{6}$$

To present our main results, we need to define the following integral operator $F: C(J,\mathbb{R})\to C(J,\mathbb{R})$ as follows:

$$\begin{aligned} \phi x(t) &= \frac{-1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,x(\tau))\,d\tau + \frac{\beta_1+t}{(\beta_1+1-\beta_2)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau,x(\tau))\,d\tau \\ &\quad -\frac{\beta_2(\beta_1+t)}{(\beta_1+1-\beta_2)\Gamma(\alpha-1)}\int_0^\eta (\eta-\tau)^{\alpha-2} f(\tau,x(\tau))\,d\tau. \end{aligned} \tag{7}$$

3 Main Results

For the forthcoming analysis, we need the following assumptions:

(A1): There exists a constant $k>0$ such that
$|f(t,x)-f(t,y)|\le k|x-y|$, for each $t\in J$ and all $x,y\in\mathbb{R}$.

(A2): The function $f: J\times\mathbb{R}\to\mathbb{R}$ is continuous.

(A3): There exists a constant $N>0$, such that $|f(t,x)|\le N$, for each $t\in J$ and all $x\in\mathbb{R}$.

Our first result is given by:

Theorem 3.1 *Assume that the hypothesis (A1) holds.*

If

$$k\frac{(\mid\beta_1+1-\beta_2\mid+\mid\beta_1+1\mid+\alpha\eta\mid\beta_2\beta_1+\beta_2\mid)}{\Gamma(\alpha+1)\mid\beta_1+-1\beta_2\mid}<1, \tag{8}$$

then the problem (1) has a unique solution on J.

Proof: We shall prove that ϕ is contraction mapping on $C(J,\mathbb{R})$.
Let $x,y\in C(J,\mathbb{R})$. Then, for each $t\in J$, we can write

$$
\begin{aligned}
\mid \ \phi x(t)-\phi y(t) \mid =\mid & \frac{-1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) \\
& + \frac{\beta_1+t}{(\beta_1+1-\beta_2)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau))\, d\tau \\
& - \frac{\beta_2(\beta_1+t)}{(\beta_1+1-\beta_2)\Gamma(\alpha-1)} \int_0^\eta (\eta-\tau)^{\alpha-2} f(\tau, x(\tau))\, d\tau \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau))\, d\tau \\
& - \frac{\beta_1+t}{(\beta_1+1-\beta_2)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau))\, d\tau \\
& + \frac{\beta_2(\beta_1+t)}{(\beta_1+1-\beta_2)\Gamma(\alpha-1)} \int_0^\eta (\eta-\tau)^{\alpha-2} f(\tau, x(\tau))\, d\tau \\
+ \frac{\beta_2(\beta_1+t)}{(\beta_1+1-\beta_2)\Gamma(\alpha-1)} & \int_0^\eta (\eta-\tau)^{\alpha-2} f(\tau, x(\tau))\, d\tau \mid .
\end{aligned} \tag{9}
$$

Thanks to $(A1)$, we obtain

$$
\begin{aligned}
|\phi(x)(t)-\phi(y)(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \mid f(\tau, x(\tau)) - f(\tau, y(\tau)) \mid d\tau \\
& + \frac{\mid \beta_1+t \mid}{\mid \beta_1+1-\beta_2 \mid \Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \mid f(\tau, x(\tau)) - f(\tau, y(\tau)) \mid d\tau \\
& + \frac{\mid \beta_2\beta_1+\beta_2 t \mid}{\mid \beta_1+1-\beta_2 \mid \Gamma(\alpha-1)} \int_0^\eta (\eta-\tau)^{\alpha-2} \mid f(\tau, x(\tau)) - f(\tau, y(\tau)) \mid d\tau \\
\leq & \frac{k \parallel x-y \parallel}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau + \frac{\mid \beta_1+1 \mid k \parallel x-y \parallel}{\mid \beta_1+1-\beta_2 \mid \Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} d\tau \\
& + \frac{\mid \beta_2\beta_1+\beta_2 \mid k \parallel x-y \parallel}{\mid \beta_1+1-\beta_2 \mid \Gamma(\alpha-1)} \int_0^\eta (\eta-\tau)^{\alpha-2} d\tau \\
\leq & k \frac{(\mid \beta_1+1-\beta_2 \mid + \mid \beta_1+1 \mid + \alpha\eta \mid \beta_2\beta_1+\beta_2 \mid)}{\Gamma(\alpha+1) \mid \beta_1+1-\beta_2 \mid} \parallel x-y \parallel .
\end{aligned} \tag{10}
$$

Therefore,

$$
\parallel \phi(x)-\phi(y) \parallel \leq k \frac{(\mid \beta_1+1-\beta_2 \mid + \mid \beta_1+1 \mid + \alpha\eta \mid \beta_2\beta_1+\beta_2 \mid)}{\Gamma(\alpha+1)} \parallel x-y \parallel . \tag{11}
$$

Thanks to (8), we conclude that ϕ is a contraction mapping. Hence by Banach fixed point theorem, there exists a unique fixed point $x \in C(J,\mathbb{R})$, which is a solution of (1).
The second result is the following.

Theorem 3.2 *Suppose that the conditions* $(A2)$ *and* $(A3)$ *are satisfied. Then the problem (1) has at least a solution on J.*

Proof: We use Scheafer's fixed point theorem to prove that ϕ has at least a fixed point on $C(J,\mathbb{R})$:

Step1: The operator ϕ is continuous on $C(J,\mathbb{R})$: Let x_n be a sequence such that $x_n \to x$ in $C(J,\mathbb{R})$. Then, for each $t \in J$, we have:

$$\begin{aligned}
|\phi(x_n)(t)-\phi(x)(t)| =| & \frac{-1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,x_n(\tau))\,d\tau \\
& +\frac{\beta_1+t}{(\beta_1+1-\beta_2)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau,x_n(\tau))\,d\tau \\
& -\frac{\beta_2(\beta_1+t)}{(\beta_1+1-\beta_2)\Gamma(\alpha-1)}\int_0^\eta (\eta-\tau)^{\alpha-2} f(\tau,x_n(\tau))\,d\tau \\
& +\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,x(\tau))\,d\tau \qquad (12) \\
& -\frac{\beta_1+t}{(\beta_1+1-\beta_2)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau,x(\tau))\,d\tau \\
& +\frac{\beta_2(\beta_1+1)}{(\beta_1+1-\beta_2)\Gamma(\alpha-1)}\int_0^\eta (\eta-\tau)^{\alpha-2} f(\tau,x(\tau))\,d\tau\ | .
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\phi(x_n)(t)-\phi(x)(t)| \leq & \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,x_n)-f(\tau,x)|d\tau \\
& +\frac{|\beta_1+1|}{|\beta_1+1-\beta_2|\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}|f(\tau,x_n)-f(\tau,x)|d\tau \qquad (13) \\
& +\frac{|\beta_2\beta_1+\beta_2|}{|\beta_1+1-\beta_2|\Gamma(\alpha-1)}\int_0^\eta (\eta-\tau)^{\alpha-2}|f(\tau,x_n)-f(\tau,x)|d\tau.
\end{aligned}$$

Thanks to $(A2)$, we obtain

$$\parallel \phi(x_n) - \phi(x) \parallel \to 0, n \to \infty. \tag{14}$$

Step2: The operator ϕ maps bounded sets into bounded sets in $C(J, \mathbb{R})$:
So, let us take $x \in B_\mu = \{x \in C(J, \mathbb{R}) ; \parallel x \parallel \leq \mu, \mu > 0\}$.
By (A3), we have:

$$\begin{aligned}
| \phi x(t) | &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} | f(\tau, x(\tau)) | d\tau \\
&+ \frac{| \beta_1 + 1 |}{| \beta_1 + 1 - \beta_2 | \Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} | f(\tau, x(\tau)) | d\tau \\
&+ \frac{| \beta_2\beta_1 + \beta_2 |}{| \beta_1 + 1 - \beta_2 | \Gamma(\alpha-1)} \int_0^\eta (\eta-\tau)^{\alpha-2} | f(\tau, x(\tau)) | d\tau
\end{aligned}$$

$$\leq \frac{N}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau + \frac{N | \beta_1 + 1 |}{| \beta_1 + 1 - \beta_2 | \Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} d\tau \tag{15}$$

$$\begin{aligned}
&+ \frac{N | \beta_2\beta_1 + \beta_2 |}{| \beta_1 + 1 - \beta_2 | \Gamma(\alpha-1)} \int_0^\eta (\eta-\tau) d\tau \\
&\leq \frac{N}{\Gamma(\alpha+1)} + \frac{N | \beta_1 + 1 |}{\Gamma(\alpha+1) | \beta_1 + 1 - \beta_2 |} + \frac{\alpha\eta N | \beta_2\beta_1 + \beta_2 |}{\Gamma(\alpha+1)}.
\end{aligned}$$

Thus,

$$\parallel \phi(x) \parallel \leq \frac{N | \beta_1 + 1 - \beta_2 | + N | \beta_1 + 1 | + \alpha\eta N | \beta_2\beta_1 + \beta_2 |}{\Gamma(\alpha+1) | \beta_1 + 1 - \beta_2 |}. \tag{16}$$

And consequently,

$$\parallel \phi(x) \parallel < \infty. \tag{17}$$

Step3: The operator ϕ maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$:
Let $t_1, t_2 \in J; t_2 < t_1, x \in B_\mu$. Then, we have

$$\begin{aligned}
| \phi x(t_1) - \phi x(t_2) | &= | \frac{-1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau \\
&+ \frac{\beta_1 + t_1}{(\beta_1 + 1 - \beta_2) \Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau \\
&- \frac{\beta_2(\beta_1 + t_1)}{(\beta_1 + 1 - \beta_2) \Gamma(\alpha-1)} \int_0^\eta (\eta-\tau)^{\alpha-2} f(\tau, x(\tau)) d\tau
\end{aligned} \tag{18}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-\tau)^{\alpha-1}f(\tau,x(\tau))\,d\tau$$
$$-\frac{\beta_1+t_2}{(\beta_1+1-\beta_2)\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}f(\tau,x(\tau))$$

$$+\frac{\beta_2(\beta_1+t_2)}{(\beta_1+1-\beta_2)\Gamma(\alpha-1)}\int_0^\eta(\eta-\tau)^{\alpha-2}f(\tau,x(\tau))\,d\tau\mid.$$

Therefore,

$$\mid\phi x(t_1)-\phi x(t_2)\mid\leq\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_1-\tau)^{\alpha-1}-(t_2-\tau)^{\alpha-1}\mid f(\tau,x(\tau))\mid d\tau$$

$$+\frac{t_1-t_2}{\mid\beta_1+1-\beta_2\mid\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\mid f(\tau,x(\tau))\mid d\tau \tag{19}$$
$$+\frac{\mid\beta_2\mid(t_1-t_2)}{\mid\beta_1+1-\beta_2\mid\Gamma(\alpha-1)}\int_0^\eta(\eta-\tau)\mid f(\tau,x(\tau))\mid d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}(t_1-\tau)^{\alpha-1}\mid f(\tau,x(\tau))\mid d\tau.$$

Thus,

$$\begin{aligned}\mid\phi x(t_1)-\phi x(t_2)\mid&\leq\frac{(N+N\alpha\eta\mid\beta_2\mid)}{\Gamma(\alpha+1)\mid\beta_1+1-\beta_2\mid}(t_1-t_2)\\&+\tfrac{2N}{\Gamma(\alpha+1)}(t_1-t_2)^\alpha+\tfrac{N}{\Gamma(\alpha+1)}(t_2^\alpha-t_1^\alpha).\end{aligned}\tag{20}$$

As $t_2\to t_1$, the right-hand side of (20) tends to zero. Then, combining the Steps 1,2,3 with Arzela-Ascoli theorem, we conclude that ϕ is completely continuous.

Step4: The set

$$\Omega=\{x\in C(J,\mathbb{R}),x=\lambda\phi(x),0<\lambda<1\}\tag{21}$$

is bounded:

Let $x\in\Omega$, then $x=\lambda\phi(x)$, for some $0<\lambda<1$. Hence, for $t\in J$, we have:

$$x(t)=\lambda[\frac{-1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau,x(\tau))+\frac{\beta_1+t}{(\beta_1+1-\beta_2)\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}f(\tau,x(\tau))\,d\tau$$

$$-\frac{\beta_2(\beta_1+t)}{(\beta_1+1-\beta_2)\Gamma(\alpha-1)}\int_0^{\eta}(\eta-\tau)^{\alpha-2}f(\tau,x(\tau))\,d\tau]. \tag{22}$$

Thanks to (A3), we can write

$$\frac{1}{\lambda}|x(t)|\leq\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\mid f(\tau,x(\tau))\mid d\tau$$

$$+\tfrac{|\beta_1+1|}{|\beta_1+1-\beta_2|\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\mid f(\tau,x)\mid d\tau$$

$$+\frac{\mid\beta_2\beta_1+\beta_2\mid}{\mid\beta_1+1-\beta_2\mid\Gamma(\alpha-1)}\int_0^{\eta}(\eta-\tau)^{\alpha-2}\mid f(\tau,x(\tau))\mid d\tau. \tag{23}$$

Therefore,

$$\parallel x\parallel\leq\lambda\frac{N(\mid\beta_1+1-\beta_2\mid+\mid\beta_1+1\mid+\alpha\eta\mid\beta_2\beta_1+\beta_2\mid)}{\Gamma(\alpha+1)\mid\beta_1+1-\beta_2\mid}. \tag{24}$$

Hence,

$$\parallel\phi(x)\parallel<\infty. \tag{25}$$

This shows that the set Ω is bounded.
As consequence of Schaefer's fixed point theorem, we deduce that ϕ has at least a fixed point, which is a solution of (1).
Now, we use Krasnselskii theorem [11] to prove the following result:

Theorem 3.3 *Assume that the hypotheses (A1)-(A2)-(A3) hold, such that*

$$k<\Gamma(\alpha+1). \tag{26}$$

If there exists $\sigma\in\mathbb{R}$ such that

$$\frac{N(\mid\beta_1+1\mid+\alpha\eta\mid\beta_2\beta_1+\beta_2\mid)}{\mid\beta_1+1-\beta_2\mid\Gamma(\alpha+1)}\leq\sigma, \tag{27}$$

then the problem (1) has at least a solution on J.

Proof: Suppose that (27) holds and let us take

$$\phi x(t):=H_1x(t)+H_2x(t), \tag{28}$$

where

$$H_1x(t):=\frac{-1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau,x(\tau))\,d\tau \tag{29}$$

and

$$H_2x\left(t\right) := \frac{\beta_1+t}{\left(\beta_1+1-\beta_2\right)\Gamma\left(\alpha\right)}\int_0^1\left(1-\tau\right)^{\alpha-1}f\left(\tau,x\left(\tau\right)\right)d\tau$$
$$-\frac{\beta_2(\beta_1+t)}{(\beta_1+1-\beta_2)\Gamma(\alpha-1)}\int_0^\eta\left(\eta-\tau\right)^{\alpha-2}f\left(\tau,x\left(\tau\right)\right)d\tau. \tag{30}$$

(1*): We shall prove that H_1 is a contraction mapping: Let $x,y\in C\left(J,\mathbb{R}\right)$. Then, for each $t\in J$, we can write

$$\mid H_1x\left(t\right)-H_1y\left(t\right)\mid=\mid -\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-\tau\right)^{\alpha-1}f\left(\tau,x\left(\tau\right)\right)d\tau$$
$$+\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-\tau\right)^{\alpha-1}f\left(\tau,y\left(\tau\right)\right)d\tau \tag{31}$$
$$\leq \frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-\tau\right)^{\alpha-1}\mid f\left(\tau,x\left(\tau\right)\right)-f\left(\tau,y\left(\tau\right)\right)\mid d\tau.$$

By (A1), we get

$$\parallel H_1\left(x\right)-H_1\left(y\right)\parallel\leq\frac{k}{\Gamma\left(\alpha+1\right)}\parallel x-y\parallel. \tag{32}$$

Using the condition (26) we conclude that H_1 is a contraction mapping.
(2*): We shall prove that H_2 is continuous:
Let x_n be sequence such that $x_n\to x$ in $C\left(J,\mathbb{R}\right)$. Then for each $t\in J$,

$$\mid H_2x_n(t)-H_2x(t)\mid=\mid\frac{\beta_1+t}{\left(\beta_1+1-\beta_2\right)\Gamma\left(\alpha\right)}\int_0^1\left(1-\tau\right)^{\alpha-1}f\left(\tau,x_n\left(\tau\right)\right)\mid$$
$$-\frac{\beta_2\left(\beta_1+t\right)}{\left(\beta_1+1-\beta_2\right)\Gamma\left(\alpha-1\right)}\int_0^\eta\left(\eta-\tau\right)^{\alpha-1}f\left(\tau,x_n\left(\tau\right)\right)d\tau$$
$$-\frac{\beta_1+t}{\left(\beta_1+1-\beta_2\right)\Gamma\left(\alpha\right)}\int_0^1\left(1-\tau\right)^{\alpha-1}f\left(\tau,x\left(\tau\right)\right)d\tau$$
$$+\frac{\beta_2\left(\beta_1+t\right)}{\left(\beta_1+1-\beta_2\right)\Gamma\left(\alpha-1\right)}\int_0^\eta\left(\eta-\tau\right)^{\alpha-1}f\left(\tau,x\left(\tau\right)\right)d\tau\mid$$

$$\leq\frac{\mid\beta_1+1\mid}{\mid\beta_1+1-\beta_2\mid\Gamma\left(\alpha\right)}\int_0^1\left(1-\tau\right)^{\alpha-1}\mid f\left(\tau,x_n\left(\tau\right)\right)-f\left(\tau,x\left(\tau\right)\right)\mid d\tau \tag{33}$$

$$\frac{\mid\beta_2\beta_1+\beta_2\mid}{\mid\beta_1+1-\beta_2\mid\Gamma\left(\alpha-1\right)}\int_0^\eta\left(\eta-\tau\right)^{\alpha-2}\mid f\left(\tau,x_n\left(\tau\right)\right)-f\left(\tau,x\left(\tau\right)\right)\mid d\tau$$

$$\leq \frac{\mid \beta_1 + 1 \mid + \alpha\eta \mid \beta_2\beta_1 + \beta_2 \mid}{\mid \beta_1 + 1 - \beta_2 \mid \Gamma(\alpha+1)} \parallel f(.,x_n) - f(.,x) \parallel$$

Since f is a continuous function, we have

$$\parallel H_2(x_n) - H_2(x) \parallel \to 0, n \to \infty. \tag{34}$$

(3*): Now, we prove that H_2 maps bounded sets into bounded sets of $C(J,\mathbb{R})$: Let $x \in B_\sigma$. It is clear that

$$\mid H_2x(t) \mid = \mid \frac{\beta_1 + t}{(\beta_1 + 1 - \beta_2)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau))\, d\tau$$
$$- \frac{\beta_2(\beta_1 + t)}{(\beta_1 + 1 - \beta_2)\Gamma(\alpha - 1)} \int_0^\eta (\eta - \tau)^{\alpha-1} f(\tau, x(\tau))\, d\tau \mid . \tag{35}$$

Therefore,

$$\mid H_2x(t) \mid \leq \frac{\mid \beta_1 + 1 \mid}{\mid \beta_1 + 1 - \beta_2 \mid \Gamma(\alpha)} \int_0^1 (1-\tau) \mid f(\tau, x(\tau)) \mid d\tau$$
$$+ \frac{\mid \beta_2\beta_1 + \beta_2 \mid}{\mid \beta_1 + 1 - \beta_2 \mid \Gamma(\alpha - 1)} \int_0^\eta (\eta - \tau)^{\alpha-2} \mid f(\tau, x(\tau)) \mid d\tau \tag{36}$$

Tanks to (A3), we can write:

$$\parallel H_2(x) \parallel \leq \frac{N \mid \beta_1 + 1 \mid}{\mid \beta_1 + 1 - \beta_2 \mid \Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} d\tau \tag{37}$$
$$+ \frac{N \mid \beta_2\beta_1 + \beta_2 \mid}{\mid \beta_1 + 1 - \beta_2 \mid \Gamma(\alpha - 1)} \int_0^\eta (\eta - \tau)^{\alpha-2} d\tau$$

Thus,

$$\parallel H_2(x) \parallel \leq \frac{N \mid \beta_1 + 1 \mid + \alpha\eta N \mid \beta_2\beta_1 + \beta_2 \mid}{\mid \beta_1 + 1 - \beta_2 \mid \Gamma(\alpha+1)}. \tag{38}$$

Consequently,

$$\parallel H_2(x) \parallel < \infty. \tag{39}$$

(4*): The operator H_2 maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$:

Let $t_1, t_2 \in J; t_1 < t_2, x \in B_\sigma$. Then, we have

$$\mid H_2x(t_2) - H_2x(t_1) \mid = \mid \frac{\beta_1 + t_2}{(\beta_1 + 1 - \beta_2)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau))\, d\tau$$

$$
\begin{aligned}
&-\frac{\beta_2\left(\beta_1+t_2\right)}{\left(\beta_1+1-\beta_2\right)\Gamma\left(\alpha-1\right)}\int_0^\eta\left(\eta-\tau\right)^{\alpha-2}f\left(\tau,x\left(\tau\right)\right)d\tau \qquad (40)\\
&-\frac{\beta_1+t_1}{\left(\beta_1+1-\beta_2\right)\Gamma\left(\alpha\right)}\int_0^1\left(1-\tau\right)^{\alpha-1}f\left(\tau,x\left(\tau\right)\right)d\tau\\
&+\frac{\beta_2\left(\beta_1+t_1\right)}{\left(\beta_1+1-\beta_2\right)\Gamma\left(\alpha-1\right)}\int_0^\eta\left(\eta-\tau\right)^{\alpha-2}f\left(\tau,x\left(\tau\right)\right)d\tau \mid .
\end{aligned}
$$

This implies that,

$$
\begin{aligned}
\mid H_2x\left(t_2\right)-H_2x\left(t_1\right)\mid\leq&\frac{t_2-t_1}{\mid\beta_1+1-\beta_2\mid\Gamma\left(\alpha\right)}\int_0^1\left(1-\tau\right)^{\alpha-1}\mid f\left(\tau,x\left(\tau\right)\right)\mid d\tau\\
&+\frac{\mid\beta_2\mid t_2-t_1}{\mid\beta_1+1-\beta_2\mid\Gamma\left(\alpha-1\right)}\int_0^\eta\left(\eta-\tau\right)^{\alpha-2}\mid f\left(\tau,x\left(\tau\right)\right)\mid d\tau. \qquad (41)
\end{aligned}
$$

Hence, we have

$$
\mid H_2x\left(t_2\right)-H_2x\left(t_1\right)\mid\leq\frac{N}{\mid\beta_1+1-\beta_2\mid\Gamma\left(\alpha+1\right)}\left(t_2-t_1\right)+\frac{\alpha\eta N\mid\beta_2\mid}{\mid\beta_1+1-\beta_2\mid\Gamma\left(\alpha+1\right)}\left(t_2-t_1\right). \qquad (42)
$$

As $t_1\to t_2$ the right-hand side of (42) tends to zero. Then, as a consequence of steps $(2*,3*,4*)$, we can conclude that H_2 is continuous and compact.

(5*): Now, we shall prove that for any $x,y\in B_\sigma$, then $H_1\left(x\right)+H_2(y)\in B_\sigma$. So, let us take $x,y\in B_\sigma$. We have:

$$
\begin{aligned}
\mid\ H_1x\left(t\right)+H_2y\left(t\right)\mid=&\mid-\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-\tau\right)^{\alpha-1}f\left(\tau,x\left(\tau\right)\right)d\tau\\
&+\frac{\beta+t}{\left(\beta_1+1-\beta_2\right)\Gamma\left(\alpha\right)}\int_0^1\left(1-\tau\right)^{\alpha-1}f\left(\tau,x\left(\tau\right)\right)d\tau\\
&-\frac{\beta_2\left(\beta_1+t\right)}{\left(\beta_1+1-\beta_2\right)\Gamma\left(\alpha-1\right)}\int_0^\eta\left(\eta-\tau\right)^{\alpha-2}f\left(\tau,x\left(\tau\right)\right)d\tau\mid\\
&\leq\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-\tau\right)^{\alpha-1}\mid f\left(\tau,x\left(\tau\right)\right)\mid d\tau \qquad (43)\\
&+\frac{\mid\beta_1+1\mid}{\mid\beta_1+1-\beta_2\mid\Gamma\left(\alpha\right)}\int_0^1\left(1-\tau\right)^{\alpha-1}\ \mid\ f\left(\tau,x\left(\tau\right)\right)\mid d\tau\\
&+\frac{\mid\beta_2\beta_1+\beta_2\mid}{\mid\beta_1+1-\beta_2\mid\Gamma\left(\alpha-1\right)}\int_0^\eta\left(\eta-\tau\right)^{\alpha-2}\ \mid\ f\left(\tau,x\left(\tau\right)\right)\mid d\tau.
\end{aligned}
$$

By (A3), we have

$$\| \quad H_1(x) + H_2(y) \| \leq \frac{N \mid \beta_1 + 1 \mid}{\mid \beta_1 + 1 - \beta_2 \mid \Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} d\tau \tag{44}$$
$$+ \frac{N \mid \beta_2\beta_1 + \beta_2 \mid}{\mid \beta_1 + 1 - \beta_2 \mid \Gamma(\alpha - 1)} \int_0^\eta (\eta - \tau)^{\alpha-2} d\tau.$$

Consequently,

$$\| H_1(x) + H_2(y) \| \leq \frac{N(\mid \beta_1 + 1 \mid + \alpha\eta \mid \beta_2\beta_1 + \beta_2 \mid)}{\mid \beta_1 + 1 - \beta_2 \mid \Gamma(\alpha + 1)}. \tag{45}$$

Using the condition (27), we conclude that $H_1(x) + H_2(y) \in B_\sigma$. As a consequence of Krasnoselskii's fixed point theorem we deduce that ϕ has a fixed point which is a solution of (1).

4 Open Problems

At the end, we pose the following open problems:
Open Problem 1: Using Riemann-Liouville fractional differential operator of order α, under what conditions do Theorems $6, 7$ and Theorem 8 hold for $1 < \alpha < 2$?
Open Problem 2: Is it possible to generalize the above results for (1), where the derivative D^α is taken in the sense of Riemann-Liouville, $n < \alpha < n+1, n \in \mathbb{N}$, and using Riemann-Liouville fractional initial conditions?

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