Int. J. Open Problems Compt. Math., Vol. 6, No. 2, June 2013
ISSN 1998-6262; Copyright ©ICSRS Publication, 2013
www.i-csrs.org

On a New Weighted Erdös-Mordell Type Inequality

Wei-Dong Jiang

Department of Information Engineering, Weihai Vocational College
Weihai 264210, ShanDong province, P. R. China
e-mail: jackjwd@163.com

Abstract

In this short note, a new weighted Erdös-Mordell inequality Involving Interior Point of a triangle is established. By it's application, some interesting geometric inequalities are derived.

Keywords: *Erdös-Mordell inequality, Geometric inequality, Triangle, spread angle theorem.*

2010 Mathematics Subject Classification: 26D15.

1 Introduction

Throughout the paper we assume $\triangle ABC$ be a Triangle, and denote by a, b, c its sides' lengths, Δ be the area. Let P be an interior point, Extend AP, BP, CP respectively to meet the opposite sides at D, E and F. Let $PD = r_1', PE = r_2', PF = r_3'$, Δ_1, Δ_2, Δ_3 denote the areas of $\triangle BPC$, $\triangle CPA$, $\triangle APB$. R_a, R_b, R_c the circumradii of the triangles BPC, CPA, APB, respectively. Let R_1, R_2, R_3 be the distances from P to A, B, C, and also let r_1, r_2, r_3 be the distances from P to the sides AB, BC, CA.

Then Erdös-Mordell inequality is true:

Theorem 1.1.

$$R_1 + R_2 + R_3 \geq 2\left(r_1 + r_2 + r_3\right) \tag{1}$$

whereat equality holds if and only if the triangle is equailateral and the point P is its center. This inequality was conjectured by Erdös in 1935[1], and was first proved by Mordell in 1937[2].

In the paper[3], D. S. Mitrnović etc noted some generalizations of Erdös-Mordell inequality in 1989. Among their results are the following theorem for three-variable quadratic Erdös-Mordell type inequality :

Theorem 1.2. *If x, y, z are three real numbers, then for any point P inside the triangle ABC, we have*

$$x^2R_1 + y^2R_2 + z^2R_3 \geq 2\left(yzr_1 + zxr_2 + xyr_3\right) \tag{2}$$

with equality holding if and only if $x = y = z$ and P is the center of equilateral $\triangle ABC$.

Recently, Jiang [6] presented a new weighted Erdös-Mordell type inequality. In this note, we give another new weighted Erdös-Mordell type inequality, as application, some interesting geometric inequalities are also established.

2 Main results

In order to prove Theorem 2.2 below, we need the following lemma.

Lemma 2.1. *For any point P inside $\triangle ABC$, $x, y, z \in R$, then we have*

$$x^2sin^2A + y^2sin^2B + z^2sin^2C \leq \frac{1}{4}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{zy}{z}\right)^2. \tag{3}$$

Proof. We make use of Kooi's inequality [4]:

For real numbers $\lambda_1, \lambda_2, \lambda_3$ with $\lambda_1 + \lambda_2 + \lambda_3 \neq 0$,

$$(\lambda_1 + \lambda_2 + \lambda_3)^2R^2 \geq \lambda_2\lambda_3a^2 + \lambda_3\lambda_1b^2 + \lambda_1\lambda_2c^2; \tag{4}$$

Where R be circumradius of triangle ABC, equality holds if and only if the point with homogeneous barycentric coordinates $(\lambda_1 : \lambda_2 : \lambda_3)$ with reference to triangle ABC is the circumcenter of the triangle.

Now, Lemma 2.1 follows from (4) with $\lambda_1 = \frac{yz}{x}, \lambda_2 = \frac{zx}{y}, \lambda_3 = \frac{xy}{z}$, and the law of sines: $a = 2R\sin A, b = 2R\sin B, c = 2R\sin C$. $\square$

Now we are in a position to state and prove our main result.

Theorem 2.2. *For any point P inside triangle ABC, Extend AP, BP, CP respectively to meet the opposite sides at D, E and F. Let R_a, R_b, R_c the circumradiuses of triangles $\triangle BPC$, $\triangle CPA$, $\triangle APB$, and let $PD = r_1', PE = r_2', PF = r_3'$. x, y, z are positive real numbers, we have*

$$\frac{xr_1'}{\sqrt{R_bR_c}} + \frac{yr_2'}{\sqrt{R_cR_a}} + \frac{zr_3'}{\sqrt{R_aR_b}} \leq \frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right). \tag{5}$$

with equality holding if and only if $x = y = z$ and P is the center of equilateral $\triangle ABC$.

Proof. Let $\angle BPC = \alpha, \angle CPA = \beta, \angle APB = \gamma$. It is obvious that $0 < \alpha, \beta, \gamma < \pi$ and $\alpha + \beta + \gamma = 2\pi$.

By using spread angle theorem, we have.

$$\begin{aligned}\frac{\sin\alpha}{r_1'} &= \frac{\sin(\pi-\beta)}{R_2} + \frac{\sin(\pi-\gamma)}{R_3}\\ &= \frac{\sin\beta}{R_2} + \frac{\sin\gamma}{R_3}\\ &\geq 2\sqrt{\frac{\sin\beta\sin\gamma}{R_2R_3}},\end{aligned}$$

Thus,

$$2r_1' \leq \sqrt{R_2R_3\csc\beta\csc\gamma}\sin\alpha.$$

Make use of $b = 2R_b\sin\beta, c = 2R_c\sin\gamma$, we get

$$\begin{aligned}\frac{r_1'}{\sqrt{R_bR_c}} &\leq \sqrt{\frac{R_2R_3}{bc}}\sin\alpha,\\ &= \sqrt{\frac{\Delta_1}{\Delta}\sin A\sin\alpha}\end{aligned}$$

Let

$$A' = \pi - \alpha, B' = \pi - \beta, C' = \pi - \gamma$$

Because

$$\sqrt{\sin A\sin\alpha} \leq \frac{1}{2}(\sin A + \sin\alpha) = \sin\frac{A+A'}{2}\cos\frac{A-A'}{2} \leq \sin\frac{A+A'}{2}$$

we have,

$$\frac{xr_1'}{\sqrt{R_bR_c}} \leq \sqrt{\frac{\Delta_1}{\Delta}}x\sin\frac{A+A'}{2}, \tag{6}$$

By the same way, one can get

$$\frac{yr_2'}{\sqrt{R_cR_a}} \leq \sqrt{\frac{\Delta_2}{\Delta}}y\sin\frac{B+B'}{2}, \tag{7}$$

$$\frac{zr_3'}{\sqrt{R_aR_b}} \leq \sqrt{\frac{\Delta_3}{\Delta}}z\sin\frac{C+C'}{2}, \tag{8}$$

Combining expression (6), (7), (8) and By Cauchy's inequality, we have

$$\begin{aligned}\sum \frac{xr_1'}{\sqrt{R_bR_c}} &\le \sum \sqrt{\frac{\Delta_1}{\Delta}} x \sin\frac{A+A'}{2}\\ &\le \sqrt{\sum \frac{\Delta_1}{\Delta} \sum x^2 \sin^2 \frac{A+A'}{2}},\\ &= \sqrt{\sum x^2 \sin^2 \frac{A+A'}{2}}.\end{aligned}$$

Let

$$\theta = \frac{A+A'}{2}, \phi = \frac{B+B'}{2}, \varphi = \frac{C+C'}{2}.$$

Obviously, $0 < \theta, \phi, \varphi < \pi$ and $\theta + \phi + \varphi = \pi$, so θ, ϕ, φ can be angles of a triangle $A_1B_1C_1$. Applying Lemma 2.1 for the triangle $A_1B_1C_1$ we obtain

$$x^2 \sin^2\theta + y^2 \sin^2\phi + z^2 \sin^2\varphi \le \frac{1}{4}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{zy}{z}\right)^2.$$

This conclude that

$$\frac{xr_1'}{\sqrt{R_bR_c}} + \frac{yr_2'}{\sqrt{R_cR_a}} + \frac{zr_3'}{\sqrt{R_aR_b}} \le \frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right).$$

and with equality holding if and only if $x = y = z$, and P is the center of equilateral $\triangle ABC$. The proof of Theorem 2.2 is completed. □

3 Some application

In this section we give some applications of Theorem 2.2.

Noticed $r_1 \le r_1'$ etc, we have

$$\frac{xr_1}{\sqrt{R_bR_c}} + \frac{yr_2}{\sqrt{R_cR_a}} + \frac{zr_3}{\sqrt{R_aR_b}} \le \frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right). \tag{9}$$

By using AM-GM inequality, we have $\sqrt{R_bR_b} \le \frac{1}{2}(R_b + R_b)$, then from (5) we have

$$\frac{xr_1'}{R_b+R_c} + \frac{yr_2'}{R_c+R_a} + \frac{zr_3'}{R_a+R_b} \le \frac{1}{4}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right). \tag{10}$$

By the same way of (9), the following inequality holds.

$$\frac{xr_1}{R_b+R_c} + \frac{yr_2}{R_c+R_a} + \frac{zr_3}{R_a+R_b} \le \frac{1}{4}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right). \tag{11}$$

let $x=y=z=1$ in (11), we have

$$\frac{r_1}{R_b+R_c}+\frac{r_2}{R_c+R_a}+\frac{r_3}{R_a+R_b}\leq\frac{3}{4}. \tag{12}$$

In fact, (12) was conjectured by Liu in [5] and here we obtained a proof.

Corollary 3.1. *If $x,y,z>0$, then*

$$x^2R_a+y^2R_b+z^2R_c\geq 2\left(yzr_1'+zxr_2'+xyr_3'\right) \tag{13}$$

Proof. alter $x\to x'\sqrt{R_bR_c}, y\to y'\sqrt{R_cR_a}, z\to z'\sqrt{R_aR_b}(x,y,z>0)$ in (5), we obtain

$$x'r_1'+y'r_2'+zr_3'\leq\frac{1}{2}\left(\frac{y'z'}{x'}R_a+\frac{z'x'}{y'}R_b+\frac{x'y'}{z'}R_c\right). \tag{14}$$

and then, let $\frac{y'z'}{x'}=x^2$, $\frac{z'x'}{y'}=y^2$, , $\frac{x'y'}{z'}=z^2$ in (14), then (13) is obtained. □

(13) is similar to (2), that was conjectured by Liu in [5].
Obviously.

$$x^2R_a+y^2R_b+z^2R_c\geq 2\left(yzr_1+zxr_2+xyr_3\right). \tag{15}$$

Let $x=y=z=1$ in (13)and (15), then we have.

$$R_a+R_b+R_c\geq 2\left(r_1'+r_2'+r_3'\right) \tag{16}$$

and

$$R_a+R_b+R_c\geq 2\left(r_1+r_2+r_3\right). \tag{17}$$

Note that (17) is similar to (1).
let $x=y=z=1$ in (5) and by AM-GM inequality, we have

$$R_aR_bR_c\geq 8r_1'r_2'r_3'. \tag{18}$$

and

$$R_aR_bR_c\geq 8r_1r_2r_3. \tag{19}$$

4 Open problem

At the end, we pose an open problem.

Open problem: For an interior point P and positive real numbers x,y,z, Let $AD=w_1', BE=w_2', CF=w_3'$, R and r denote the circumradius and inradius of triangle ABC respectively, then

$$\frac{xw_1'}{\sqrt{R_bR_c}}+\frac{yw_2'}{\sqrt{R_cR_a}}+\frac{zw_3'}{\sqrt{R_aR_b}}\leq\sqrt{2+\frac{r}{2R}}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right). \tag{20}$$

ACKNOWLEDGEMENTS. This work is supported, in part, by the Project of Shandong Province Higher Educational Science and Technology Program under grant No. J11LA57.

References

[1] P. Erdos, *Problem 3740.* Amer. Math. Monthly, **42** (1935), pp. 396.

[2] L. J. Mordell AND D. F. Barrow. *Solution 3740*, Amer. Math. Monthly. **44**(1937), 252-254.

[3] D. S. Mitrinoviĉ, J. E. Pečarć and V. Volonеć. *Recent Advances in Geometric Inequalities. Dordrecht*, Netherlands: Kluwer. 1989.

[4] O. Bottema etc. *Geometric Inequalities.* Groningen: Wolters-Noordhoff. 1969.

[5] Jian Liu. *A hundred unsolved Triangle inequality problem*, geometric inequalities in china. nanjing: Jiangsu education Press, 1996, (in Chinese).

[6] Wei-Dong Jiang, *A new weighted Erdös-Mordell type inequality*, Creative Math. & Inf. **17**(2008): 152-155.