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A Monotonicity Method in Quasistatic Processes for Viscoplastic Materials of the from $\dot{\sigma} = \mathcal{E}(\varepsilon(\dot{u}), \theta) + F(\sigma, \varepsilon(u), \chi, \theta)$

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Abstract

In this paper, we study a quasistatic problem for semi-linear rate-type viscoplastic models with two parameters κ, θ ***may be interpreted as the absolute temperature or an internal state variable. The existence and uniqueness of the solution is proved using monotony arguments followed by a Cauchy-Lipschitz technique.***

Keywords: *Strongly monotone, viscoplastic, Existence and uniqueness, Lipschitz operator.*

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1 Introduction

Throught this paper, we consider Ω as a bounded domain in $\mathbb{R}^N$ $(N = 1, 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and Γ_1 is an open subset of Γ such that meas $\Gamma_1 > 0$. We denote by $\Gamma_2 = \Gamma - \overline{\Gamma}_1$. Let ν be the outward unit normal vector, on Γ and S_N the set of second order symmetric tensors on R^N. Let T be a real positive constant and M a natural number.

Consider the problem

$$\dot{\sigma} = \mathcal{E}(\varepsilon(\dot{u}),\theta) + F(\sigma,\varepsilon(u),\kappa,\theta) \ \text{ in } \ \Omega\times(0,T), \tag{1}$$
$$\dot{\kappa} = \varphi(\sigma,\varepsilon(u),\kappa,\theta) \ \text{ in } \ \Omega\times(0,T), \tag{2}$$
$$Div\ \sigma + f = \ 0 \ \text{ in } \ \Omega\times(0,T), \tag{3}$$
$$u = g \ \text{ on } \ \Gamma_1\times(0,T), \tag{4}$$
$$\sigma\nu = h \ \text{ on } \ \Gamma_2\times(0,T), \tag{5}$$
$$u(0) = u_0, \ \ \sigma(0) = \sigma_0, \ \ \kappa(0) = \kappa_0 \ \text{ in } \ \Omega. \tag{6}$$

In it problem the unknowns are the displacement function $u : \Omega\times[0,T] \to R^N$, the stress function $\sigma : \Omega\times[0,T] \to S_N$ and the internal state variable $\kappa : \Omega\times[0,T] \to R^M$.
This problem represents a quasi-static problem for rate-type models of the form (1), (2) in which κ may be interpreted as an internal state variable and θ is a parameter where $\mathcal{E}$ is a non linear function, $\varepsilon(u) : \Omega\times[0,T] \to S_N$ is the small strain tensor (i.e $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^t u)$). In (1) and (2) $\mathcal{E}$, F and φ are given constitutive functions. In (3) $Div\ \sigma$ represents the divergence of vector valued function σ, f represents given body force, g and h are the given boundary data, and, finally, u_0, σ_0, κ_0 are the initial data.

Initial and boundary value problems for models of the form (1), (2) for differents forms $\mathcal{E}$, F and φ were studied by Djabi. So, existence and uniqueness results were given by Djabi [2] (the case when $\mathcal{E}$ depends on $\varepsilon(\dot{u})$ and F,φ depend on $(\sigma,\varepsilon(u),\kappa\ )$.

In the case when $\mathcal{E}$ depends on $(\varepsilon(\dot{u}),\kappa)$ and F,φ depend on $(\sigma,\varepsilon(u),\kappa)$; existence and uniquenes results concerning the problem (1)-(2) were obtained by Djabi [1] using montony arguments followed by a Cauchy-Lipschitz technique.
The purpose of this paper is to prove the existence and uniqueness of the solution for the problem (1)-(6) when $\mathcal{E}$ is a nonlinear function and F,φ depending on $\sigma,\varepsilon(u),\kappa$ and θ, by using monotony arguments followed by a Cauchy-Lipschitz technique (Theorem 3.1).

2 Preliminaries

Everywhere in this paper we utilise the following notations: ".“ the inner product on the spaces $\mathbb{R}^N$, $\mathbb{R}^M$ and S_N and $|\cdot|$ are the Euclidean norms on these spaces.

$$H \ = \{ \ v = (v_i) \mid v_i \in L^2(\Omega), \ \ i = \overline{1,N} \ \},$$

$$H_1 = \{ \ v = (v_i) \mid v_i \in H^1(\Omega), \ \ i = \overline{1,N} \ \},$$

$$\mathcal{H} = \{ \ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega), \quad i,j = \overline{1,N} \ \},$$

$$\mathcal{H}_1 = \{ \ \tau = (\tau_{ij}) \mid Div \ \tau \in H \ \},$$
$$Y = \{ \ \kappa = (\kappa_i) \mid \kappa_i \in L^2(\Omega), \quad i = \overline{1,M} \ \}.$$

The spaces H, H_1, $\mathcal{H}$, $\mathcal{H}_1$ and Y are real Hilbert spaces endowed with the canonical inner products denoted by $< \cdot,\cdot >_H$, $< \cdot,\cdot >_{H_1}$, $< \cdot,\cdot >_{\mathcal{H}}$, $< \cdot,\cdot >_{\mathcal{H}_1}$ and $< \cdot,\cdot >_Y$ respectively.

Let $H_\Gamma = [H^{\frac{1}{2}}(\Gamma)]^N$ and $\gamma : H_1 \to H_\Gamma$ be the trace map. We denote by

$$V = \{ \ u \in H_1 \mid \gamma u = 0 \text{ on } \ \Gamma_1 \ \},$$

and let E be the subspace of H_Γ defined by

$$E = \gamma(V) = \{ \ \xi \in H_\Gamma \mid \xi = 0 \ \text{ on } \Gamma_1 \}.$$

Let $H'_\Gamma = [H^{-\frac{1}{2}}(\Gamma)]^N$ be the strong dual of the space H_Γ and let $< \cdot,\cdot >$ denote the duality between H'_Γ and H_Γ. If $\tau \in \mathcal{H}_1$ there exists an element $\gamma_\nu \tau \in H'_\Gamma$ such that

$$< \gamma_\nu \tau, \gamma v > = < \tau, \varepsilon(v) >_{\mathcal{H}} + < Div \ \tau, v >_H \quad \text{for all } v \in H_1. \tag{7}$$

By $\tau\nu$ we shall understand the element of E' (the strong dual of E) that is the projection of $\gamma_\nu \tau$ on E.

Let us now denote by $\mathcal{V}$ the following subspace of $\mathcal{H}_1$.

$$\mathcal{V} = \{ \ \tau \in \mathcal{H}_1 \mid \ Div \ \tau = 0 \ \text{ in } \ \Omega, \quad \tau\nu = 0 \ \text{ on } \ \Gamma_2 \ \}.$$

Using (7), it may be proved that $\varepsilon(V)$ is the orthogonal complement of $\mathcal{V}$ in $\mathcal{H}$, hence

$$< \tau, \varepsilon(v) >_{\mathcal{H}} = 0 \ \text{ for all } \ v \in V \ , \quad \tau \in \mathcal{V}. \tag{8}$$

Finally, for every real Hilbert space X we denote by $|\cdot|_X$ the norm on X and by $C^j(0,T,X)$ $(j = 0,1)$ the spaces defined as follows :

$C^0(0,T,X) = \{z : [0,T] \to X \mid z \ \text{ is continuous } \}$, Let us recall that if $C^j(0,T,X)$ are real Banach spaces endowed with the norms

$C^1(0,T,X) = \{z : [0,T] \to X \mid$ there exists $\dot{z}$ the derivate of z and $\dot{z} \in C^0(0,T,X)\}$.

$$|z|_{0,T,X} = \max_{t\in[0,T]} |z(t)|_X, \tag{9}$$

and

$$|z|_{1,T,X} = |z|_{0,T,X} + |\dot{z}|_{0,T,X}.$$

respectively.

Let us recall that if K is a convex closed non empty set of X and $P : X \to K$ is the projector map on K, we have

$$y = Px \text{ if and only if } y \in K \ \text{ and } \ < y - x, \ z - x >_X \geq 0 \text{ for all } z \in K. \tag{10}$$

3 Main results

In the study of the problem (1)-(6), we consider the following assumptions:

$$\begin{cases} \mathcal{E}:\Omega\times S_N\times\mathbb{R}^p\rightarrow S_N \text{ and} \\ \text{(a) there exists } m>0 \text{ such that } <\mathcal{E}(\varepsilon_1,\theta_1)-\mathcal{E}(\varepsilon_2,\theta_2),\varepsilon_1-\varepsilon_2>\geq \\ \qquad\qquad \geq m|\varepsilon_1-\varepsilon_2|^2 \text{ for all } \varepsilon_1,\varepsilon_2\in S_N, \theta\in\mathbb{R}^p \text{ a.e. in } \Omega, \\ \text{(b) there exists } L'>0 \text{ such that } |\mathcal{E}(\varepsilon_1,\theta_1)-\mathcal{E}(\varepsilon_2,\theta_2)|\leq L'|\varepsilon_1-\varepsilon_2| \\ +|\theta_1-\theta_2| \text{ for all } \varepsilon_1,\varepsilon_2\in S_N, \theta_1,\theta_2\in\mathbb{R}^p, \text{a.e. in } \Omega, \\ \text{(c) } x\rightarrow\mathcal{E}(x,\varepsilon,\theta) \text{ is a measurable function with respect to} \\ \text{the lebesgue measure in } \Omega \text{ for all } \varepsilon\in S_N\ ,\ \theta\in\mathbb{R}^p \\ \text{(d) } x\rightarrow\mathcal{E}(x,0,0)\in\mathcal{H}. \end{cases} \tag{11}$$

$$\begin{cases} F:\Omega\times S_N\times S_N\times\mathbb{R}^M\times\mathbb{R}^P\rightarrow S_N \text{ and} \\ \text{a) there exists } L>0 \text{ such that} \\ \qquad |F(x,\sigma_1,\varepsilon_1,\kappa_1,\theta_1)-F(x,\sigma_2,\varepsilon_2,\kappa_2,\theta_2)|\leq \\ \qquad \leq\ L\left(|\sigma_1-\sigma_2|+|\varepsilon_1-\varepsilon_2|+|\kappa_1-\kappa_2|+|\theta_1-\theta_2|\right) \\ \text{for all } \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2\in S_N,\ \kappa_1,\kappa_2\in\mathbb{R}^M,\ \theta_1,\theta_2\in\mathbb{R}^P, \text{a.e. in } \Omega, \\ \text{(b) } x\rightarrow F(x,\sigma,\varepsilon,\kappa,\theta) \text{ is a measurable function with respect to} \\ \text{the Lebesgue measure on } \Omega,\ \text{for all } \sigma,\varepsilon\in S_N,\ \kappa\in\mathbb{R}^M,\ \theta\in\mathbb{R}^P, \\ \text{(c) } x\rightarrow F(x,0,0,0,0)\in\mathcal{H}. \end{cases} \tag{12}$$

$$\begin{cases} \varphi:\Omega\times S_N\times S_N\times\mathbb{R}^M\times\mathbb{R}^P\rightarrow\mathbb{R}^M \text{ and} \\ \text{(a) there exists } L'>0 \text{ such that} \\ \qquad |\varphi(x,\sigma_1,\varepsilon_1,\kappa_1,\theta_1)-\varphi(x,\sigma_2,\varepsilon_2,\kappa_2,\theta_2)|\leq \\ \qquad \leq\ L'\left(|\sigma_1-\sigma_2|+|\varepsilon_1-\varepsilon_2|+|\kappa_1-\kappa_2|+|\theta_1-\theta_2|\right) \\ \text{for all } \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2\in S_N,\kappa_1,\kappa_2\in\mathbb{R}^M,\theta_1,\theta_2\in\mathbb{R}^P, \text{a.e.in } \Omega, \\ \text{(b) } x\rightarrow\varphi(x,\sigma,\varepsilon,\kappa,\theta) \text{ is a measurable function with respect to} \\ \text{the Lebesgue measure on } \Omega,\ \text{for all } \sigma,\varepsilon\in S_N,\ \kappa\in\mathbb{R}^M,\ \theta\in\mathbb{R}^P, \\ \text{(c) } x\rightarrow\varphi(x,0,0,0,0)\in Y. \end{cases} \tag{13}$$

$$f\ \in C^1\left(0,T,H\right),\quad g\in\left(0,T,H_\Gamma\right),\quad h\in C^1\left(0,T,E'\right). \tag{14}$$

$$\mathcal{K}_0\in Y. \tag{15}$$

$$u_0\in H,\quad \sigma_0\in\mathcal{H}_1. \tag{16}$$

$$Div\ \sigma_0+f\left(0\right)=0 \text{ in } \Omega,\ u_0=g\left(0\right) \text{ on } \Gamma_1,\ \sigma_0\nu=h\left(0\right) \text{ on } \Gamma_2. \tag{17}$$

$$\theta\in C^0\left(0,T,L^2\left(\Omega\right)^P\right). \tag{18}$$

The main result of this section is as follows.

Theorem 3.1 *Let (11)-(18) hold. Then there exists a unique solution* $u \in C^1(0,T,H_1), \sigma \in C^1(0,T,\mathcal{H}_1), \kappa \in C^1(0,T,Y)$ *of the problem (1)-(6).*

In order to prove Theorem 3.1, we need some preliminaries.

Let $\tilde{u} \in C^1(0,T,H_1)$, $\tilde{\sigma} \in C^1(0,T,\mathcal{H}_1)$ be two functions such that

$$Div\ \tilde{\sigma} + f = 0 \quad \text{in} \ \ \Omega \times (0,T), \tag{19}$$

$$\tilde{u} = g \quad \text{on} \ \ \Gamma_1 \times (0,T), \tag{20}$$

$$\tilde{\sigma}\nu = h \quad \text{on} \ \ \Gamma_2 \times (0,T). \tag{21}$$

The existence of this couple follows from (14) and the properties of the trace maps.

Considering the functions defined by

$$\bar{u} = \ u - \tilde{u}, \quad \bar{\sigma} = \sigma - \tilde{\sigma}, \tag{22}$$

$$\bar{u}_0 = \ u_0 - \tilde{u}_0, \quad \bar{\sigma}_0 = \sigma_0 - \tilde{\sigma}_0. \tag{23}$$

It easy to see that the triplet $(u,\sigma,\kappa) \in C^1\left(0,T,H \times \mathcal{H}_1 \times L^2(\Omega)^M\right)$ is a solution of the problem (1)-(6) if and only if

$$(\bar{u},\bar{\sigma},\kappa) \in C^1\left(0,T,V \times \mathcal{V} \times L^2(\Omega)^M\right), \tag{24}$$

$$\dot{\bar{\sigma}} = \mathcal{E}(\varepsilon(\dot{\bar{u}}) + \varepsilon(\dot{\tilde{u}}),\theta) + F(\bar{\sigma}+\tilde{\sigma},\varepsilon(\bar{u})+\varepsilon(\tilde{u}),\kappa,\theta) - \dot{\tilde{\sigma}} \text{ in } \Omega \times (0,T), \tag{25}$$

$$\bar{u}(0) = \ \bar{u}_0, \quad \bar{\sigma}(0) = \bar{\sigma}_0 \text{ in } \ \Omega, \tag{26}$$

$$\dot{\kappa} = \varphi(\bar{\sigma}+\tilde{\sigma},\varepsilon(\bar{u})+\varepsilon(\tilde{u}),\kappa,\theta) \ \text{ in } \Omega \times (0,T), \tag{27}$$

$$\kappa(0) = \kappa_0 \text{ in } \ \Omega. \tag{28}$$

To solve the problem (24)-(28), we consider the product Hilbert spaces $X = \varepsilon(V) \times \left\{0_{L^2(\Omega)}\right\}$, $Z = \mathcal{V} \times L^2(\Omega)^P$, $H = \mathcal{H} \times L^2(\Omega)^M$, $Z' = X \times Z$ and the operators $S, G, \mathcal{F}$ defined by

$$S : L^2(\Omega)^P \times \varepsilon(V) \times \mathcal{V} \times \mathcal{H} \times L^2(\Omega)^M \to \varepsilon(V) \times \left\{0_{L^2(\Omega)}\right\},$$
$$S = P \circ \mathcal{F},$$

where P is the projector map on $\varepsilon(V) \times \left\{0_{L^2(\Omega)}\right\}$ and

$$\mathcal{F}(\theta,x',y',z') = [G(\theta,x,y,z,r),\tilde{\varphi}(\theta,x,y,r)] \tag{29}$$

for all

$x' = (x, 0) \in X, y' = (y, r) \in Z, z' = (z, \mu) \in \mathcal{H} \times L^2(\Omega)^M, \theta \in L^2(\Omega)^P$, where

$$G(\theta, x, y, z, r) = \mathcal{E}(z + \varepsilon(\tilde{u}(t), \theta(t)) + F(y + \tilde{\sigma}(t), x + \varepsilon(\tilde{u}(t)), r(t), \theta(t)) - \tilde{\sigma}(t), \tag{30}$$

$$\tilde{\varphi}(\theta, x, y, r) = \varphi(y + \tilde{\sigma}(t), x + \varepsilon(\tilde{u}(t)), r(t), \theta(t))? \tag{31}$$

We have the following result.

Lemma 3.2 *Let $\theta \in L^2(\Omega)^P$, $x' \in X$ and $y' \in Z$.*
Then there exists a unique element $z' = (q', r') \in Z'$ such that

$$\tau' = \mathcal{F}(\theta, x', y', z'). \tag{32}$$

Proof. The uniqueness part is a consequence of (11); indeed, if $z_1' = (q_1', \tau_1'),\ \ z_2' = (q_2', \tau_2') \in Z'$ are such that

$$\begin{aligned} \tau_1' = \mathcal{F}(\theta, x', y', z_1') = [G(\theta, x, y, z_1, r), \tilde{\varphi}(\theta, x, y, r)], \\ \tau_2' = \mathcal{F}(\theta, x', y', z_2') = [G(\theta, x, y, z_2, r), \tilde{\varphi}(\theta, x, y, r)], \end{aligned}$$

where

$$\begin{gathered} G(\theta, x, y, z_i, r) = \mathcal{E}(z_i + \varepsilon(\tilde{u}(t), \theta(t)))+ \\ F(y + \tilde{\sigma}(t), x + \varepsilon(\tilde{u}(t)), r(t), \theta(t)) - \tilde{\sigma}(t),\ (\ i = 1, 2). \end{gathered}$$

Then using (11.a), we have

$$\begin{gathered} \langle \tau_1' - \tau_2',\ z_1 - z_2 \rangle_{\mathcal{H} \times L^2(\Omega)^M} = \\ \langle \mathcal{E}(z_1 + \varepsilon(\tilde{u}(t), \theta(t))) - \mathcal{E}(z_2 + \varepsilon(\tilde{u}(t), \theta(t))), z_{1-} z_2 \rangle_{\mathcal{H} \times L^2(\Omega)^M} \geq \\ m\, |z_{1-} z_2|^2_{\mathcal{H} \times L^2(\Omega)^M}. \end{gathered}$$

Using now the orthogonality in $\mathcal{H} \times L^2(\Omega)^M$ of $(\tau_1' - \tau_2') \in \mathcal{V} \times L^2(\Omega)^M$ and $(z_1 - z_2) \in \varepsilon(V) \times L^2(\Omega)^M$,we deduce that $z_1 = z_2$, which implies $\tau_1' = \tau_2'$.

For the existence part, using the hypotheses on $\mathcal{E}$,G, φ and the properties of the projectors, we can prove, for t, x', y' fixed the following inequalites:

$$\left\{ \begin{gathered} \langle S(\theta, x', y', z_1') - S(\theta, x', y', z_2'), z_1' - z_2' \rangle_{\mathcal{H} \times L^2(\Omega)^M} \geq \\ \geq \langle \mathcal{F}(\theta, x', y', z_1') - \mathcal{F}(\theta, x', y', z_2'), z_1' - z_2' \rangle_{\mathcal{H} \times L^2(\Omega)^M} \geq \\ \geq m\, |z_1' - z_2'|^2_{\mathcal{H} \times L^2(\Omega)^M}. \end{gathered} \right. \tag{33}$$

Moreover, from (11), (12), (13) and the properties of the projectors, we get

$$\left\{ \begin{gathered} \langle S(\theta, x', y', z_1') - S(\theta, x', y', z_2'), z_1' - z_2' \rangle_{\mathcal{H} \times L^2(\Omega)^M} \leq \\ \leq |\mathcal{F}(\theta, x', y', z_1') - \mathcal{F}(\theta, x', y', z_2')|_{\mathcal{H} \times L^2(\Omega)^M} \leq \\ \leq L'\, |q_1 - q_2|^2_{\mathcal{H} \times L^2(\Omega)^M}. \end{gathered} \right. \tag{34}$$

Hence $S(\theta, x', y', .) : \varepsilon(V) \times \left\{0_{L^2(\Omega)^M}\right\} \to \varepsilon(V) \times \left\{0_{L^2(\Omega)^M}\right\}$ is a strongly monotome Lipschitz operator. Using now Browder's surjectivity theorem we get that there exists $q' \in \varepsilon(V) \times \{0_{L^2(\Omega)^M}\}$ $S(\theta, x', y', q') = 0_{\varepsilon(V)\times\{0_{L^2(\Omega)^M}\}}$. It results that the element $\mathcal{F}(\theta, x', y', q')$ belongs to $\mathcal{V} \times L^2(\Omega)^M$ and we finish the proof using $z' = (q', \tau')$ where

$$\tau' = \mathcal{F}(\theta, x', y', q') = [F(\theta, x, y, z, r), \tilde{\varphi}(\theta, x, y, r)].$$

We consider now the operator $A : L^2(\Omega)^P \times Z' \to Z'$ defined as follows:

$$\begin{cases} A(\theta, \omega') = z' \\ \omega' = (x', y'), z' = (q', \tau') \\ \tau' = \mathcal{F}(\theta, x', y', q'). \end{cases} \tag{35}$$

We have

Lemma 3.3 *For all $\theta \in L^2(\Omega)^P$ and $\omega_1', \omega_2' \in Z'$, the operator $A : L^2(\Omega)^P \times Z' \to Z'$ is continuous and there exists $C > 0$ such that*

$$|A(\theta, \omega_1') - A(\theta, \omega_2')|_{Z'} \leq C|\omega_1' - \omega_2'|_{Z'} \text{ for all } \theta \in L^2(\Omega)^P,\ \omega_1',\ \omega_2' \in Z'. \tag{36}$$

Proof. Let $\theta_i \in L^2(\Omega)^P$, $\omega_i' = (x_i', y_i') \in Z'$ and $z_i' = (q_i', \tau_i') = A(\theta_i, \omega_i')$, $i = 1, 2$. Then (29) implies

$$S(\theta_i, x_i', y_i', q_i') = 0_{\varepsilon(V)} \times \{0_{L^2(\Omega)^M}\},\ i = 1, 2. \tag{37}$$

Using the hypothesies on $\mathcal{E}$, F, φ and the proprieties of the projectors, we get:

$$\begin{aligned} m|q_1' - q_2'|^2_{\mathcal{H}} &\leq < S(\theta_1, x_1, y_1, q_1') - S(\theta_1, x_1, y_1, q_2'), \varepsilon(v_1) - \varepsilon(v_2) >_{\mathcal{H}} \\ &= < S(\theta_2, x_2', y_2', q_2') - S(\theta_1, x_1', y_1', q_1'), q_1' - q_2' >^2_{\mathcal{H}\times L^2(\Omega)^M} \\ &\leq |\mathcal{F}(\theta_2, x_2', y_2', q_2') - \mathcal{F}(\theta_1, x_1', y_1', q_1')|_{\mathcal{H}\times L^2(\Omega)^{M^2}} \times |q_1' - q_2'|^2_{\mathcal{H}\times L^2(\Omega)^M}. \end{aligned}$$

Which implies

$$|q_1' - q_2'|_{\mathcal{H}\times L^2(\Omega)^M} \leq \frac{1}{m} \times |\mathcal{F}(\theta_1, x_1', y_1', q_2') - \mathcal{F}(\theta_2, x_2', y_2', q_2')|_{\mathcal{H}\times L^2(\Omega)^M}. \tag{38}$$

Using now (29), (30), (31) and (32), we get

$$|\tau_1' - \tau_2'|_{\mathcal{H}\times L^2(\Omega)^M} = |\mathcal{F}(\theta_1, x_1', y_1', q_1') - \mathcal{F}(\theta_2, x_2', y_2', q_2')|_{\mathcal{H}\times L^2(\Omega)^M} \tag{39}$$

Hence

$$\begin{cases} |\tau_1' - \tau_2'|_{\mathcal{H}\times L^2(\Omega)^M} \leq L'|q_1' - q_2'|_{\mathcal{H}\times L^2(\Omega)^M} + \\ |\mathcal{F}(\theta_1, x_1', y_1', q_1') - \mathcal{F}(\theta_2, x_2', y_2', q_2')|_{\mathcal{H}\times L^2(\Omega)^M} \end{cases} \tag{40}$$

Then it results

$$\begin{cases} |\tau_1' - \tau_2'|_{\mathcal{H}\times L^2(\Omega)^M} \leq \\ (\frac{L'}{m}+1)|\mathcal{F}(\theta_1, x_1', y_1', q_1') - \mathcal{F}(\theta_2, x_2', y_2', q_2')|_{\mathcal{H}\times L^2(\Omega)^M} \end{cases} \tag{41}$$

Using (34) we get

$$|\mathcal{F}(\theta_1, x_1', y_1', q_1') - \mathcal{F}(\theta_2, x_2', y_2', q_2')|_{\mathcal{H}\times L^2(\Omega)^M} \leq L(|x_1' - x_2'| + |y_1' - y_2'|) + \\ |\mathcal{F}(\theta_1, x_1', y_1', q_1') - \mathcal{F}(\theta_2, x_2', y_2', q_2')|_{\mathcal{H}\times L^2(\Omega)^M}.$$

Using (40), we have

$$|A(\theta_1, \omega_1') - A(\theta_2, \omega_2')|_Z \leq \\ \frac{1}{m}|\mathcal{F}(\theta_1, x_1', y_1', q_1') - \mathcal{F}(\theta_2, x_2', y_2', q_2')|_{\mathcal{H}\times L^2(\Omega)^M} \\ +(\frac{L'}{m}+1)|\mathcal{F}(\theta_1, x_1', y_1', q_1') - \mathcal{F}(\theta_2, x_2', y_2', q_2')|_{\mathcal{H}\times L^2(\Omega)^M}. \tag{42}$$

Hence, by $\theta \to \mathcal{F}(\theta, x', y', q')$; $L^2(\Omega)^p \to X \oplus Y$ is an continuous operator, for all $x' \in X$, $y' \in Y$, $z' \in H$.

We obtain

$$|\mathcal{F}(\theta_1, x_1', y_1', q_1') - \mathcal{F}(\theta_2, x_2', y_2', q_2')|_{\mathcal{H}\times L^2(\Omega)^M} \to 0.$$

when $\theta_1 \to \theta_2$ in $L^2(\Omega)^P$, $x_1' \to x_2'$ in X, $y_1' \to y_2'$ in Z.

Thus, we obtain that A is continuous operator. Taking $\theta_1 = \theta_2$ in (41) it results

$$\begin{cases} |A(\theta_1, \omega_1') - A(\theta_2, \omega_2')|_Z \leq C|\omega_1' - \omega_2'| \text{ forall } \theta_1 \in L^2(\Omega)^P, \\ \omega_1', \omega_2' \in Z'. \end{cases} \tag{43}$$

Proof of Theorem 3.1.

Using the definition of operator A, we get that $\bar{u}, \bar{\sigma}, k$ is solution to (24)-(28), if and only if

$$z = ((\varepsilon(\bar{u}), 0), (\bar{\sigma}, k)) \in C^1(0, T, Z')$$

and

$$\dot{z}' = (\dot{x}', \dot{y}') = A(\theta, z'(\theta)) \text{ for all } \theta_2 \in L^2(\Omega)^P \tag{44}$$

$$z'(0) = z_0 = ((\varepsilon(\bar{u}_0), 0), (\bar{\sigma}_0, k_0)). \tag{45}$$

In order to study the problem (43)-(44), let us remark that, by Lemma 3.3, A is a continuous operator and

$$|A(\theta_1, z_1') - A(\theta_2, z_2')|_{Z'} \leq C|z_1' - z_2'|_{Z'} \text{ for all } \theta \in L^2(\Omega)^P \text{ and } z_1', z_2' \in Z'.$$

Let $B : [0.T] \times Z' \to Z'$ and $z_0^{'}$ be defind by

$$\begin{cases} B(t, z') = A(\theta(t), z') \text{ for all } t \in [0.T] \text{ and } z_0' \in Z'. \\ \qquad z_0' = (x_0', y_0')\,. \end{cases} \tag{46}$$

and

$$\begin{aligned} z'(0) &= (x'(0), y'(0)) = ((x(0), 0), (y(0), 0)) \\ &= ((\varepsilon(\bar{u}_0), 0), (\bar{\sigma}_0, k_0)) \in X \times Y = Z'. \end{aligned}$$

Using the definition of A, we get that

$x' \in C^1(0, T, X)$ and $y' \in C^1(0, T, Y)$ is a solution of (24)-(28), if and only if $z' = (x', y') \in C^1(0, T, X \times Y)$ is a solution of the problem

$$\dot{z}'(t) = B(t, z'(t)) \text{ for all } [0, T]\,, \tag{47}$$
$$z'(0) = z_0^{'}. \tag{48}$$

where

$$B(t, z^{'}(t)) = A(\theta(t), z^{'}(t)),\ z^{'} = (x^{'}, y^{'})\,, \quad y' = \mathcal{F}(\theta, x', y', q'),$$

In order to study the problem (44)-(45), let us remark that, by Lemma 3.3 and $\theta \in C^1(0, T, L^2(\Omega)^P)$, we get that B is a continuous operator,and

$$|B(t, z_1') - B(t, z_2')|_{Z'} \leq C|z_1' - z_2'|_{Z'} \text{ for all } t \in [0.T] \text{ and } z_1', z_2' \in Z'.$$

Moreover, by (21) and (22), $\tilde{u} \in C^1(0, T, H_1)$ and $\tilde{\sigma} \in C^1(0, T, \mathcal{H}_1)$.

We that $z_0^{'}$ belongs to Z' and by Lemma 3.2 and the classical Cauchy-Lipschitz theorem we have that $z' \in C^1(0, T, Z')$ and the proof of Theorem 3.1 is complete.

4 Open Problem

The case when the dissipative function φ in the differential equation governed by the second internal state variable κ is not necessary Lipschitzian (for example, the viscose dissipation in the energy conservation equation, which can be written as the product of the stress tensor and the plastic part of the rate of deformation tensor) remains unsolved and need several mathematical techniques, like expansive fixed point theorems and other arguments.

We noticed that if we admit that the first parameter θ, which represents the thermal effects, becomes an internal state variable, the process may depend also on the energy conservation equation. That problem has been not studied in this work. In addition it is well-known that this situation leads to thermal instability.

Moreover, it is of interest to investigate setting with taking into account the phenomena of contact with or without friction. Mathematically, these are likely to turn out to be vey hard problems. There is the possibility of non existence or non uniqueness of solutions.

We also notice that the processes of dynamic evolution for these rate-type constitutive laws have been never treated. New mathematical tools need to be developed for this task. Since variational methods are incapable to solve these problems, we must use numerical techniques to approximate and simulate such models.

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