A Monotonicity Method in Quasistatic Processes for Viscoplastic Materials of the form \( \dot{\sigma} = \mathcal{E}(\varepsilon(\dot{u}), \theta) + F(\sigma, \varepsilon(u), \chi, \theta) \)

F. Messelmi\(^1\) and Abdelbaki. Merouani\(^2\)

\(^1\) Faculty of Sciences and Technology, University of Djelfa, Djelfa, P.O.Box 3117 Algeria
\(^2\) Department of mathematics, Bordj Bou Arreridj University, Algeria

Abstract

In this paper, we study a quasistatic problem for semilinear rate-type viscoplastic models with two parameters \( \kappa, \theta \) may be interpreted as the absolute temperature or an internal state variable. The existence and uniqueness of the solution is proved using monotony arguments followed by a Cauchy-Lipschitz technique.

Keywords: Strongly monotone, viscoplastic, Existence and uniqueness, Lipschitz operator.

2010 Mathematics Subject Classification: 39B82, 39B52, 39B55, 46C05, 81Q05.73C50.74M15.

1 Introduction

Throught this paper, we consider \( \Omega \) as a bounded domain in \( \mathbb{R}^N \) \((N = 1, 2, 3)\) with a smooth boundary \( \partial \Omega = \Gamma \) and \( \Gamma_1 \) is an open subset of \( \Gamma \) such that \( \text{meas } \Gamma_1 > 0 \). We denote by \( \Gamma_2 = \Gamma - \overline{\Gamma}_1 \). Let \( \nu \) be the outward unit normal vector, on \( \Gamma \) and \( S_N \) the set of second order symmetric tensors on \( \mathbb{R}^N \). Let \( T \) be a real positive constant and \( M \) a natural number.
Consider the problem

\[ \dot{\sigma} = \mathcal{E}(\varepsilon(\dot{u}), \theta) + F(\sigma, \varepsilon(u), \kappa, \theta) \quad \text{in} \quad \Omega \times (0, T), \quad (1) \]

\[ \dot{\kappa} = \varphi(\sigma, \varepsilon(u), \kappa, \theta) \quad \text{in} \quad \Omega \times (0, T), \quad (2) \]

\[ \text{Div } \sigma + f = 0 \quad \text{in} \quad \Omega \times (0, T), \quad (3) \]

\[ u = g \quad \text{on} \quad \Gamma_1 \times (0, T), \quad (4) \]

\[ \sigma \nu = h \quad \text{on} \quad \Gamma_2 \times (0, T), \quad (5) \]

\[ u(0) = u_0, \quad \sigma(0) = \sigma_0, \quad \kappa(0) = \kappa_0 \quad \text{in} \quad \Omega. \quad (6) \]

In it problem the unknowns are the displacement function \( u : \Omega \times [0, T] \rightarrow \mathbb{R}^N \), the stress function \( \sigma : \Omega \times [0, T] \rightarrow S_N \) and the internal state variable \( \kappa : \Omega \times [0, T] \rightarrow \mathbb{R}^M \).

This problem represents a quasi-static problem for rate-type models of the form (1), (2) in which \( \kappa \) may be interpreted as an internal state variable and \( \theta \) is a parameter where \( \mathcal{E} \) is a non linear function, \( \varepsilon(u) : \Omega \times [0, T] \rightarrow S_N \) is the small strain tensor (i.e \( \varepsilon(u) = \frac{1}{2} (\nabla u + \nabla^t u) \)). In (1) and (2) \( \mathcal{E}, F \) and \( \varphi \) are given constitutive functions. In (3) \( \text{Div } \sigma \) represents the divergence of vector valued function \( \sigma \), \( f \) represents given body force, \( g \) and \( h \) are the given boundary data, and, finally, \( u_0, \sigma_0, \kappa_0 \) are the initial data.

Initial and boundary value problems for models of the form (1), (2) for different forms \( \mathcal{E}, F \) and \( \varphi \) were studied by Djabi. So, existence and uniqueness results were given by Djabi [2] (the case when \( \mathcal{E} \) depends on \( \varepsilon(\dot{u}) \) and \( F, \varphi \) depend on \( (\sigma, \varepsilon(u), \kappa) \)).

In the case when \( \mathcal{E} \) depends on \( (\varepsilon(\dot{u}), \kappa) \) and \( F, \varphi \) depend on \( (\sigma, \varepsilon(u), \kappa) \); existence and uniqueness results concerning the problem (1)-(2) were obtained by Djabi [1] using monotony arguments followed by a Cauchy-Lipschitz technique.

The purpose of this paper is to prove the existence and uniqueness of the solution for the problem (1)-(6) when \( \mathcal{E} \) is a nonlinear function and \( F, \varphi \) depending on \( \sigma, \varepsilon(u), \kappa \) and \( \theta \), by using monotony arguments followed by a Cauchy-Lipschitz technique (Theorem 3.1).

2 Preliminaries

Everywhere in this paper we utilise the following notations: \( \cdot, \cdot \) the inner product on the spaces \( \mathbb{R}^N, \mathbb{R}^M \) and \( S_N \) and \( | \cdot | \) are the Euclidean norms on these spaces.

\[ H = \{ v = (v_i) \mid v_i \in L^2(\Omega), \ i = 1, N \}, \]

\[ H_1 = \{ v = (v_i) \mid v_i \in H^1(\Omega), \ i = 1, N \}. \]
\( \mathcal{H} = \{ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega), \; i, j = 1, N \} \),

\[ \mathcal{H}_1 = \{ \tau = (\tau_{ij}) \mid \text{Div } \tau \in H \}, \]

\[ Y = \{ \kappa = (\kappa_i) \mid \kappa_i \in L^2(\Omega), \; i = 1, M \}. \]

The spaces \( H, H_1, \mathcal{H}, \mathcal{H}_1 \) and \( Y \) are real Hilbert spaces endowed with the canonical inner products denoted by \( < \cdot, \cdot >_H, < \cdot, \cdot >_{H_1}, < \cdot, \cdot >_\mathcal{H}, < \cdot, \cdot >_{\mathcal{H}_1} \) and \( < \cdot, \cdot >_Y \) respectively.

Let \( H_\Gamma = [H^2(\Gamma)]^N \) and \( \gamma : H_1 \to H_\Gamma \) be the trace map. We denote by

\[ V = \{ u \in H_1 \mid \gamma u = 0 \text{ on } \Gamma_1 \}, \]

and let \( E \) be the subspace of \( H_\Gamma \) defined by

\[ E = \gamma(V) = \{ \xi \in H_\Gamma \mid \xi = 0 \text{ on } \Gamma_1 \}. \]

Let \( H'_\Gamma = [H^{-2}(\Gamma)]^N \) be the strong dual of the space \( H_\Gamma \) and let \( < \cdot, \cdot > \) denote the duality between \( H'_\Gamma \) and \( H_\Gamma \). If \( \tau \in \mathcal{H}_1 \) there exists an element \( \gamma_{\nu} \tau \in H'_\Gamma \) such that

\[ < \gamma_{\nu} \tau, \gamma v > = < \tau, \varepsilon(v) >_H + < \text{Div } \tau, v >_H \text{ for all } v \in H_1. \]  

By \( \tau \nu \) we shall understand the element of \( E' \) (the strong dual of \( E \)) that is the projection of \( \gamma_{\nu} \tau \) on \( E \).

Let us now denote by \( \mathcal{V} \) the following subspace of \( \mathcal{H}_1 \).

\[ \mathcal{V} = \{ \tau \in \mathcal{H}_1 \mid \text{Div } \tau = 0 \text{ in } \Omega, \; \tau \nu = 0 \text{ on } \Gamma_2 \}. \]

Using (7), it may be proved that \( \varepsilon(V) \) is the orthogonal complement of \( \mathcal{V} \) in \( \mathcal{H} \), hence

\[ < \tau, \varepsilon(v) >_H = 0 \text{ for all } v \in \mathcal{V}, \; \tau \in \mathcal{V}. \]  

Finally, for every real Hilbert space \( X \) we denote by \( | \cdot |_X \) the norm on \( X \) and by \( C^j(0, T, X) \) (\( j = 0, 1 \)) the spaces defined as follows :

\[ C^0(0, T, X) = \{ z : [0, T] \to X \mid z \text{ is continuous } \}, \]

Let us recall that if \( C^1(0, T, X) \) are real Banach spaces endowed with the norms

\[ C^1(0, T, X) = \{ z : [0, T] \to X \mid \text{there exists } \dot{z} \text{ the derivate of } z \text{ and } \dot{z} \in C^0(0, T, X) \}. \]

\[ |z|_{0, T, X} = \max_{t \in [0, T]} |z(t)|_X, \]

and

\[ |z|_{1, T, X} = |z|_{0, T, X} + |\dot{z}|_{0, T, X}. \]

respectively.

Let us recall that if \( K \) is a convex closed non empty set of \( X \) and \( P : X \to K \) is the projector map on \( K \), we have

\[ y = Px \text{ if and only if } y \in K \text{ and } < y - x, z - x >_X \geq 0 \text{ for all } z \in K. \]  

(10)
3 Main results

In the study of the problem (1)-(6), we consider the following assumptions:

\[
\begin{cases}
\mathcal{E} : \Omega \times S_N \times \mathbb{R}^p \to S_N \quad \text{and} \\
\quad \text{(a) there exists } m > 0 \text{ such that } < \mathcal{E}(\varepsilon_1, \theta_1) - \mathcal{E}(\varepsilon_2, \theta_2), \varepsilon_1 - \varepsilon_2 > \geq m \varepsilon_1 - \varepsilon_2^2 \text{ for all } \varepsilon_1, \varepsilon_2 \in S_N, \theta \in \mathbb{R}^p \text{ a.e. in } \Omega, \\
\quad \text{(b) there exists } L' > 0 \text{ such that } |\mathcal{E}(\varepsilon_1, \theta_1) - \mathcal{E}(\varepsilon_2, \theta_2)| \leq L'|\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| \text{ for all } \varepsilon_1, \varepsilon_2 \in S_N, \theta_1, \theta_2 \in \mathbb{R}^p, \text{ a.e. in } \Omega, \\
\quad \text{(c) } x \to \mathcal{E}(x, \varepsilon, \theta) \text{ is a measurable function with respect to the Lebesgue measure in } \Omega \text{ for all } \varepsilon \in S_N, \theta \in \mathbb{R}^p \\
\quad \text{(d) } x \to \mathcal{E}(x, 0, 0) \in \mathcal{H}.
\end{cases}
\]

\[
F : \Omega \times S_N \times S_N \times \mathbb{R}^M \times \mathbb{R}^p \to S_N \quad \text{and} \\
\begin{cases}
\text{a) there exists } L > 0 \text{ such that } \\
\quad |F(x, \sigma_1, \varepsilon_1, \kappa_1, \theta_1) - F(x, \sigma_2, \varepsilon_2, \kappa_2, \theta_2)| \leq L \left(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |\kappa_1 - \kappa_2| + |\theta_1 - \theta_2|\right) \\
\quad \text{for all } \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in S_N, \kappa_1, \kappa_2 \in \mathbb{R}^M, \theta_1, \theta_2 \in \mathbb{R}^p, \text{ a.e. in } \Omega, \\
\text{b) } x \to F(x, \sigma, \varepsilon, \kappa, \theta) \text{ is a measurable function with respect to the Lebesgue measure on } \Omega, \text{ for all } \sigma, \varepsilon, \kappa \in \mathbb{R}^M, \theta \in \mathbb{R}^p, \\
\text{c) } x \to F(x, 0, 0, 0) \in \mathcal{H}.
\end{cases}
\]

\[
\varphi : \Omega \times S_N \times S_N \times \mathbb{R}^M \times \mathbb{R}^p \to \mathbb{R}^M \quad \text{and} \\
\begin{cases}
\text{a) there exists } L' > 0 \text{ such that } \\
\quad |\varphi(x, \sigma_1, \varepsilon_1, \kappa_1, \theta_1) - \varphi(x, \sigma_2, \varepsilon_2, \kappa_2, \theta_2)| \leq L' \left(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |\kappa_1 - \kappa_2| + |\theta_1 - \theta_2|\right) \\
\quad \text{for all } \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in S_N, \kappa_1, \kappa_2 \in \mathbb{R}^M, \theta_1, \theta_2 \in \mathbb{R}^p, \text{ a.e. in } \Omega, \\
\text{b) } x \to \varphi(x, \sigma, \varepsilon, \kappa, \theta) \text{ is a measurable function with respect to the Lebesgue measure on } \Omega, \text{ for all } \sigma, \varepsilon, \kappa \in \mathbb{R}^M, \theta \in \mathbb{R}^p, \\
\text{c) } x \to \varphi(x, 0, 0, 0, 0) \in Y.
\end{cases}
\]

\[
f \in C^1(0, T, H), \quad g \in (0, T, H_T), \quad h \in C^1(0, T, E').
\]

\[
K_0 \in Y.
\]

\[
u_0 \in H, \quad \sigma_0 \in \mathcal{H}_1.
\]

\[
Div \sigma_0 + f(0) = 0 \text{ in } \Omega, \quad u_0 = g(0) \text{ on } \Gamma_1, \quad \sigma_0 \nu = h(0) \text{ on } \Gamma_2.
\]

\[
\theta \in C^0 \left(0, T, L^2(\Omega)^p\right).
\]

The main result of this section is as follows.
Theorem 3.1 Let (11)-(18) hold. Then there exists a unique solution
\[ u \in C^1(0, T, H_1), \sigma \in C^1(0, T, H_1), \kappa \in C^1(0, T, Y) \] of the problem (1)-(6).

In order to prove Theorem 3.1, we need some preliminaries.

Let \( \tilde{u} \in C^1(0, T, H_1), \tilde{\sigma} \in C^1(0, T, H_1) \) be two functions such that
\[
\begin{align*}
\text{Div} \tilde{\sigma} + f &= 0 \quad \text{in } \Omega \times (0, T), \\
\tilde{u} &= g \quad \text{on } \Gamma_1 \times (0, T), \\
\tilde{\sigma} \nu &= h \quad \text{on } \Gamma_2 \times (0, T).
\end{align*}
\]

The existence of this couple follows from (14) and the properties of the trace maps.

Considering the functions defined by
\[
\begin{align*}
\bar{u} &= u - \tilde{u}, \quad \bar{\sigma} = \sigma - \tilde{\sigma}, \\
\bar{u}_0 &= u_0 - \tilde{u}_0, \quad \bar{\sigma}_0 = \sigma_0 - \tilde{\sigma}_0.
\end{align*}
\]

It easy to see that the triplet \( (u, \sigma, \kappa) \in C^1 \left(0, T, H \times H_1 \times L^2(\Omega)^M\right) \) is a solution of the problem (1)-(6) if and only if
\[
\begin{align*}
(\bar{u}, \bar{\sigma}, \bar{\kappa}) &\in C^1 \left(0, T, V \times V \times L^2(\Omega)^M\right), \\
\bar{\sigma} &= E(\varepsilon(\bar{u}) + \varepsilon(\bar{\sigma}), \theta) + F(\bar{\sigma} + \varepsilon(\bar{u}), \varepsilon(\bar{u}), \kappa, \theta) - \bar{\sigma} \quad \text{in } \Omega \times (0, T), \\
\bar{u}(0) &= \bar{u}_0, \quad \bar{\sigma}(0) = \bar{\sigma}_0 \quad \text{in } \Omega, \\
\dot{\kappa} &= \varphi(\bar{\sigma} + \varepsilon(\bar{u}), \varepsilon(\bar{u}), \kappa, \theta) \quad \text{in } \Omega \times (0, T), \\
\kappa(0) &= \kappa_0 \quad \text{in } \Omega.
\end{align*}
\]

To solve the problem (24)-(28), we consider the product Hilbert spaces
\( X = \varepsilon(V) \times \{0_{L^2(\Omega)}\}, \ Z = V \times L^2(\Omega)^P, H = H \times L^2(\Omega)^M, Z' = X \times Z \) and the operators \( S, G, \mathcal{F} \) defined by
\[
S : L^2(\Omega)^P \times \varepsilon(V) \times V \times H \times L^2(\Omega)^M \to \varepsilon(V) \times \{0_{L^2(\Omega)}\},
\]
\[
S = P \circ \mathcal{F},
\]
where \( P \) is the projector map on \( \varepsilon(V) \times \{0_{L^2(\Omega)}\} \) and
\[
\mathcal{F}(\theta, x', y', z') = [G(\theta, x, y, z, r), \tilde{\varphi}(\theta, x, y, r)]
\]
for all.
Lemma 3.2 Let \( \theta \in L^2(\Omega)^P \), \( x' \in X \) and \( y' \in Z \).

Then there exists a unique element \( z' = (q', r') \in Z' \) such that

\[
\tau' = \mathcal{F}(\theta, x', y', z').
\]  

Proof. The uniqueness part is a consequence of (11); indeed, if \( \tau'_1 = (q'_1, \tau'_1) \), \( \tau'_2 = (q'_2, \tau'_2) \in Z' \) are such that

\[
\tau'_1 = \mathcal{F}(\theta, x', y', z'_1) = [G(\theta, x, y, z_1, r), \tilde{\varphi}(\theta, x, y, r)], \\
\tau'_2 = \mathcal{F}(\theta, x', y', z'_2) = [G(\theta, x, y, z_2, r), \tilde{\varphi}(\theta, x, y, r)],
\]

where

\[
G(\theta, x, y, z, r) = \mathcal{E}(z + \varepsilon(\tilde{u}(t), \theta(t))) + F(y + \tilde{\sigma}(t), x + \varepsilon(\tilde{u}(t)), r(t), \theta(t)) - \tilde{\sigma}(t), \quad (30)
\]

\[
\tilde{\varphi}(\theta, x, y, r) = \varphi(y + \tilde{\sigma}(t), x + \varepsilon(\tilde{u}(t)), r(t), \theta(t)). \quad (31)
\]

We have the following result.

Then using (11.a), we have

\[
\langle \tau'_1 - \tau'_2, z_1 - z_2 \rangle_{\mathcal{H} \times L^2(\Omega)^M} = \\
\langle \mathcal{E}(z_1 + \varepsilon(\tilde{u}(t), \theta(t))) - \mathcal{E}(z_2 + \varepsilon(\tilde{u}(t), \theta(t))), z_1 - z_2 \rangle_{\mathcal{H} \times L^2(\Omega)^M} \geq \\
m |z_1 - z_2|^2_{\mathcal{H} \times L^2(\Omega)^M}.
\]

Using now the orthogonality in \( \mathcal{H} \times L^2(\Omega)^M \) of \( (\tau'_1 - \tau'_2) \in \mathcal{V} \times L^2(\Omega)^M \)

and \( (z_1 - z_2) \in \mathcal{V} \times L^2(\Omega)^M \), we deduce that \( z_1 = z_2 \), which implies \( \tau'_1 = \tau'_2 \).

For the existence part, using the hypotheses on \( \mathcal{E}, \mathcal{F}, \varphi \) and the properties of the projectors, we can prove, for \( t, x', y' \) fixed the following inequalities:

\[
\begin{align*}
\langle S(\theta, x', y', z'_1) - S(\theta, x', y', z'_2), z'_1 - z'_2 \rangle_{\mathcal{H} \times L^2(\Omega)^M} & \geq \\
\mathcal{F}(\theta, x', y', z'_1) - \mathcal{F}(\theta, x', y', z'_2), z'_1 - z'_2 \rangle_{\mathcal{H} \times L^2(\Omega)^M} & \geq \\
m |z'_1 - z'_2|^2_{\mathcal{H} \times L^2(\Omega)^M}. & \quad (33)
\end{align*}
\]

Moreover, from (11), (12), (13) and the properties of the projectors, we get

\[
\begin{align*}
\langle S(\theta, x', y', z'_1) - S(\theta, x', y', z'_2), z'_1 - z'_2 \rangle_{\mathcal{H} \times L^2(\Omega)^M} & \leq \\
|\mathcal{F}(\theta, x', y', z'_1) - \mathcal{F}(\theta, x', y', z'_2)|_{\mathcal{H} \times L^2(\Omega)^M} & \leq \\
L |q_1 - q_2|^2_{\mathcal{H} \times L^2(\Omega)^M}. & \quad (34)
\end{align*}
\]
Hence \( S(\theta, x', y', \cdot) : \varepsilon(V) \times \{0_{L^2(\Omega)^M}\} \rightarrow \varepsilon(V) \times \{0_{L^2(\Omega)^M}\} \) is a strongly monotone Lipschitz operator. Using now Browder’s surjectivity theorem we get that there exists \( q' \in \varepsilon(V) \times \{0_{L^2(\Omega)^M}\} \). It results that the element \( F(\theta, x', y', q') \) belongs to \( V \times L^2(\Omega)^M \) and we finish the proof using \( z' = (q', \tau') \) where

\[
\tau' = F(\theta, x', y', q') = [F(\theta, x, y, z, r), \phi(\theta, x, y, r)].
\]

We consider now the operator \( A : L^2(\Omega)^P \times Z' \rightarrow Z' \) defined as follows:

\[
\begin{cases}
A(\theta, \omega') = z' \\
\omega' = (x', y'), z' = (q', \tau') \\
\tau' = F(\theta, x', y', q').
\end{cases}
\]

We have

**Lemma 3.3** For all \( \theta \in L^2(\Omega)^P \) and \( \omega'_1, \omega'_2 \in Z' \), the operator \( A : L^2(\Omega)^P \times Z' \rightarrow Z' \) is continuous and there exists \( C > 0 \) such that

\[
|A(\theta, \omega'_1) - A(\theta, \omega'_2)|_{Z'} \leq C|\omega'_1 - \omega'_2|_{Z'} \text{ for all } \theta \in L^2(\Omega)^P, \omega'_1, \omega'_2 \in Z'.
\]

**Proof.** Let \( \theta_i \in L^2(\Omega)^P, \omega'_i = (x'_i, y'_i) \in Z' \) and \( z'_i = (q'_i, \tau'_i) = A(\theta_i, \omega'_i), i = 1, 2 \). Then (29) implies

\[
S(\theta_i, x'_i, y'_i, q'_i) = 0_{\varepsilon(V)} \times \{0_{L^2(\Omega)^M}\}, i = 1, 2.
\]

Using the hypotheses on \( \mathcal{E}, F, \phi \) and the proprieties of the projectors, we get:

\[
m|q'_1 - q'_2|_{H \times L^2(\Omega)^M}^2 \leq \langle S(\theta_1, x_1, y_1, q'_1) - S(\theta_1, x_1, y_1, q'_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_H
\]

\[
= \langle S(\theta_2, x'_2, y'_2, q'_2) - S(\theta_1, x'_1, y'_1, q'_1), q'_1 - q'_2 \rangle_{H \times L^2(\Omega)^M}^2
\]

\[
\leq |F(\theta_2, x'_2, y'_2, q'_2) - F(\theta_1, x'_1, y'_1, q'_1)|_{H \times L^2(\Omega)^M}^2 \times |q'_1 - q'_2|_{H \times L^2(\Omega)^M}^2.
\]

Which implies

\[
|q'_1 - q'_2|_{H \times L^2(\Omega)^M} \leq \frac{1}{m} \times |F(\theta_1, x'_1, y'_1, q'_2) - F(\theta_2, x'_2, y'_2, q'_2)|_{H \times L^2(\Omega)^M}.
\]

Using now (29), (30), (31) and (32), we get

\[
|\tau'_1 - \tau'_2|_{H \times L^2(\Omega)^M} = |F(\theta_1, x'_1, y'_1, q'_1) - F(\theta_2, x'_2, y'_2, q'_2)|_{H \times L^2(\Omega)^M}
\]

Hence

\[
\begin{cases}
|\tau'_1 - \tau'_2|_{H \times L^2(\Omega)^M} \leq L'|q'_1 - q'_2|_{H \times L^2(\Omega)^M} + |F(\theta_1, x'_1, y'_1, q'_1) - F(\theta_2, x'_2, y'_2, q'_2)|_{H \times L^2(\Omega)^M}
\end{cases}
\]
Then it results

\[
\begin{cases}
|\tau'_1 - \tau'_2|_{H \times L^2(\Omega)^M} \\ (\frac{L'}{m} + 1)|\mathcal{F}(\theta_1, x'_1, y'_1, q'_1) - \mathcal{F}(\theta_2, x'_2, y'_2, q'_2)|_{H \times L^2(\Omega)^M}
\end{cases}
\] (41)

Using (34) we get

\[
|\mathcal{F}(\theta_1, x'_1, y'_1, q'_1) - \mathcal{F}(\theta_2, x'_2, y'_2, q'_2)|_{H \times L^2(\Omega)^M} \leq L(|x'_1 - x'_2| + |y'_1 - y'_2|) + |\mathcal{F}(\theta_1, x'_1, y'_1, q'_1) - \mathcal{F}(\theta_2, x'_2, y'_2, q'_2)|_{H \times L^2(\Omega)^M}.
\]

Using (40), we have

\[
|A(\theta_1, \omega'_1) - A(\theta_2, \omega'_2)|_Z \leq \frac{1}{m} |\mathcal{F}(\theta_1, x'_1, y'_1, q'_1) - \mathcal{F}(\theta_2, x'_2, y'_2, q'_2)|_{H \times L^2(\Omega)^M} + (\frac{L'}{m} + 1)|\mathcal{F}(\theta_1, x'_1, y'_1, q'_1) - \mathcal{F}(\theta_2, x'_2, y'_2, q'_2)|_{H \times L^2(\Omega)^M}.
\] (42)

Hence, by \( \theta \to \mathcal{F}(\theta, x', y', q') \); \( L^2(\Omega)^p \to X \oplus Y \) is an continuous operator, for all \( x' \in X, y' \in Y, z' \in H \).

We obtain

\[
|\mathcal{F}(\theta_1, x'_1, y'_1, q'_1) - \mathcal{F}(\theta_2, x'_2, y'_2, q'_2)|_{H \times L^2(\Omega)^M} \to 0.
\]

when \( \theta_1 \to \theta_2 \) in \( L^2(\Omega)^P \), \( x'_1 \to x'_2 \) in \( X \), \( y'_1 \to y'_2 \) in \( Z \).

Thus, we obtain that \( A \) is continuous operator. Taking \( \theta_1 = \theta_2 \) in (41) it results

\[
\begin{cases}
|A(\theta_1, \omega'_1) - A(\theta_2, \omega'_2)|_Z \leq C|\omega'_1 - \omega'_2| \text{ for all } \theta_1 \in L^2(\Omega)^P, \\
\omega'_1, \omega'_2 \in Z'.
\end{cases}
\] (43)

**Proof of Theorem 3.1.**

Using the definition of operator \( A \), we get that \( \bar{u}, \bar{\sigma}, k \) is solution to (24)-(28), if and only if

\[
z = ((\varepsilon(\bar{u}), 0), (\sigma, k)) \in C^1(0, T, Z')
\]

and

\[
\dot{z}' = (\dot{x}', \dot{y}') = A(\theta, z'(\theta)) \text{ for all } \theta_2 \in L^2(\Omega)^P, \\
\dot{z}'(0) = z_0 = ((\varepsilon(\bar{u}_0), 0), (\bar{\sigma}_0, k_0)).
\] (44) (45)
In order to study the problem (43)-(44), let us remark that, by Lemma 3.3, $A$ is a continuous operator and

$$|A(\theta_1, z'_1) - A(\theta_2, z'_2)|_{Z'} \leq C|z'_1 - z'_2|_{Z'} \text{ for all } \theta \in L^2(\Omega)P \text{ and } z'_1, z'_2 \in Z'.$$

Let $B : [0.T] \times Z' \to Z'$ and $z'_0$ be defined by

$$\begin{cases}
B(t, z') = A(\theta(t), z') \text{ for all } t \in [0.T] \text{ and } z'_0 \in Z'.
\end{cases} \tag{46}$$

and

$$
\begin{align*}
z'(0) &= (x'(0), y'(0)) = ((x(0), 0), (y(0), 0)) \\
&= ((\varepsilon(\bar{u}_0), 0), (\bar{\sigma}_0, k_0)) \in X \times Y = Z'.
\end{align*} \tag{47}$$

Using the definition of $A$, we get that

$x' \in C^1(0, T, X)$ and $y' \in C^1(0, T, Y)$ is a solution of (24)-(28), if and only if $z' = (x', y') \in C^1(0, T, X \times Y)$ is a solution of the problem

$$\begin{align*}
z'(t) &= B(t, z'(t)) \text{ for all } [0, T], \\
z'(0) &= z'_0. \tag{48}
\end{align*}$$

where

$$B(t, z'(t)) = A(\theta(t), z'(t)), \ z' = (x', y'), \ y' = \mathcal{F}(\theta, x', y', q').$$

In order to study the problem (44)-(45), let us remark that, by Lemma 3.3 and $\theta \in C^1(0, T, L^2(\Omega)P)$, we get that $B$ is a continuous operator, and

$$|B(t, z'_1) - B(t, z'_2)|_{Z'} \leq C|z'_1 - z'_2|_{Z'} \text{ for all } t \in [0.T] \text{ and } z'_1, z'_2 \in Z'.$$

Moreover, by (21) and (22), $\bar{u} \in C^1(0, T, H_1)$ and $\bar{\sigma} \in C^1(0, T, \mathcal{H}_1)$.

We that $z'_0$ belongs to $Z'$ and by Lemma 3.2 and the classical Cauchy-Lipschitz theorem we have that $z' \in C^1(0, T, Z')$ and the proof of Theorem 3.1 is complete.

4 Open Problem

The case when the dissipative function $\varphi$ in the differential equation governed by the second internal state variable $\kappa$ is not necessary Lipschitzian (for example, the viscose dissipation in the energy conservation equation, which can be
written as the product of the stress tensor and the plastic part of the rate of deformation tensor) remains unsolved and need several mathematical techniques, like expansive fixed point theorems and other arguments.

We noticed that if we admit that the first parameter $\theta$, which represents the thermal effects, becomes an internal state variable, the process may depend also on the energy conservation equation. That problem has been not studied in this work. In addition it is well-known that this situation leads to thermal instability.

Moreover, it is of interest to investigate setting with taking into account the phenomena of contact with or without friction. Mathematically, these are likely to turn out to be very hard problems. There is the possibility of non existence or non uniqueness of solutions.

We also notice that the processes of dynamic evolution for these rate-type constitutive laws have been never treated. New mathematical tools need to be developed for this task. Since variational methods are incapable to solve these problems, we must use numerical techniques to approximate and simulate such models.

References


