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Arithmetics in the set of beta-polynomials

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Abstract

Let β ***be a formal series with*** $deg(\beta) \geq 2$***, the aim of this paper is to prove that the maximal length of the finite*** β***-fractional parts in the*** β***-expansion of product of two beta-polynomials (a formal series that have not*** β***-fractional part), denoted*** $L_{\odot}(\beta)$ ***is finite when*** β ***is Pisot or Salem series. Especially, we give its exact value if*** β ***have one conjugate with absolute value smaller than*** $\frac{1}{|\beta|}$ ***and if*** β ***is a Pisot series verifying*** $\beta^d + A_{d-1}\beta^{d-1} + \cdots + A_0 = 0$ ***such that*** $\textbf{deg}(\beta) = m \geq 2$ ***and*** $\textbf{deg}(A_0) = s \geq \textbf{deg}(A_i) \quad \forall 0 \leq i \leq d-2$***.***

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1 Introduction

The β-expansion of real numbers was introduced by A. Rényi [12]. Since its introduction in 1975, its properties arithmetic, diophantine and ergodic have been extensively studied by several authors.

Let $\beta > 1$ be a real number. The β-expansion of a real number $x \in [0, 1)$ is defined by the sequence $(x_i)_{i\geq 1}$ with values in $\{0, 1, ..., [\beta]\}$ produced by the β-transformation $T_\beta :\ x \longrightarrow\ \beta x \pmod 1$ as follows :

$$\forall\ i \geq 1,\ x_i = [\beta T_\beta^{i-1}(x)],\ \text{and thus } x = \sum_{i\geq 1} \frac{x_i}{\beta^i}$$

We write $d_\beta(x) = 0.x_1x_2....$
In [11], Parry showed that for any $x \in [0,1)$, $d_\beta(x)$ is the only transformation of x in base β which satisfies the following condition called the Parry condition:

$$\forall n \in \mathbb{N}^*,\ S^n((x_i)_{i\in\mathbb{N}^*})) <_{lex} d^*_\beta(1)$$

where $S((x_i)_{i\in\mathbb{N}^*})) = (x_{i+1})_{i\in\mathbb{N}^*}$ and $d^*_\beta(1) = \begin{cases} d_\beta(1) & \text{if } d_\beta(1) \text{ is infini} \\ (t_1...t_{m-1}, t_m - 1)^\infty & \text{if } d_\beta(1) = 0.t_1...t_m. \end{cases}$

Now let $x > 1$ be a positive real number, so there exist $k \in \mathbb{N}^*$ such that $\beta^{k-1} \leq x < \beta^k$. So $\frac{x}{\beta^k} \in [0,1)$ and let $d_\beta(\dfrac{x}{\beta^k}) = (x_i)_{i\geq 1}$ finely we get $d_\beta(x) = (x_{i+k+1})_{i\geq -k}$.

Let $d_\beta(x) = (x_i)_{i\geq -n}$, so $x = \sum\limits_{i=0}^{n} x_{-i}\beta^i + \sum\limits_{i>0} x_i\beta^{-i}$. The part with non-negatives powers of β is called the β-integer part of x, denoted by $[x]_\beta$. The part with negatives powers of β is called the β-fractional part of x, denoted by $\{x\}_\beta = x - [x]_\beta$, this allows a natural generalization of the definition of development of real number in base 10.
If there exists $n \in \mathbb{N}$ such that $|x| = \sum\limits_{i=0}^{n} x_i\beta^i$, where $x_n \cdots x_0$ is the β-expansion of $|x|$, then x is called β-integer and the set of β-integers is denoted by $\mathbf{Z}_\beta$.
The set $Fin(\beta)$, introduced in [6], is defined by $\{\beta^{-k}\mathbf{Z}_\beta/k \in \mathbb{N}\}$. It allows to generalize the frame work of numeration to the case of a non-integer base β.
We know that Z_β and $Fin(\beta)$ are not stable under usual operations like addition and multiplication.
In order to study arithmetics on β-integers, we interested on the β-expansion of the number obtained by addition or multiplication of two β-integers when the β-expansion of the sum or the product is finite.
The following notation $L_\oplus(\beta)$ and $L_\odot(\beta)$ are introduced in [8].

Definition 1.1

$$L_\oplus(\beta) = min\{n \in \mathbb{N} \setminus \forall x, y \in \mathbf{Z}_\beta \ \ such\ that\ \ x + y \in Fin(\beta) \Rightarrow \beta^n(x + y) \in \mathbf{Z}_\beta\}$$

$$L_\odot(\beta) = min\{n \in \mathbb{N} \setminus \forall x, y \in \mathbf{Z}_\beta \ \ such\ that\ \ xy \in Fin(\beta) \Rightarrow \beta^n(xy) \in \mathbf{Z}_\beta\}$$

Minimum of an empty set is defined to be $+\infty$.

We can see that $L_\oplus(\beta)$ and $L_\odot(\beta)$ represent respectively the maximal possible finite length of the β-fractional part which may appear when one adds or multiplies two β-integers, otherwise they designate the maximal finite shift after the comma for the sum or product of two β-integers.
The computation of these values gives an indication on the difficulty of performing arithmetics on $\mathbf{Z}_\beta$.
Let us explain now why we are interested in the case where $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite: Indeed, if the sum or the product of two β-integers belongs to $Fin(\beta)$, then the length of the β-fractional part of this sum or product is bounded by a constant which only

depends on β.
If the set of the length sums or products of two β-integers is unbounded, then performing arithmetics in $\mathbf{Z}_\beta$ will be very difficult if not impossible, since one can not compute in a finite time any operation on β-integers.
Let us mention that to our knowledge no example is known of a β such that $L_\oplus(\beta)$ and $L_\odot(\beta)$ are infinite, however it has been proven in [6] and [7] that $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite when β is Pisot (a real algebraic integer greater than 1 with all conjugates strictly inside the unit disk). The computation of these values is however not so easy, especially for $L_\oplus(\beta)$. The case of quadratic Pisot numbers has been studied in [5] when β is a unit. The authors gave exact values for $L_\oplus(\beta)$ and $L_\odot(\beta)$, when $\beta > 1$ is a solution either of equation $x^2 = mx - 1, m \in \mathbb{N}, m \geq 3$ or of equation $x = mx + 1, m \in \mathbb{N}$. In the first case $L_\oplus(\beta) = L_\odot(\beta) = 1$, in the second case $L_\oplus(\beta) = L_\odot(\beta) = 2$, and in [8] otherwise. However, when β is of higher degree, it is a difficult problem to compute the exact value of $L_\oplus(\beta)$ or $L_\odot(\beta)$, and even to compute upper and lower bounds for these two constants.
Several examples are Studied in [2], where a method is described in order to compute upper bounds for $L_\oplus(\beta)$ and $L_\odot(\beta)$ for Pisot numbers satisfying additional algebraic properties, for example, in the Tribonacci case, that is, when β is the positive root, of the polynomial $x^3 - x^2 - x - 1$, we have $L_\oplus(\beta) = 5$, let's note that until now, we don't know the value of $L_\odot(\beta)$ in the case of the Tribonacci number, it is only proven in [2] that $4 \leq L_\odot(\beta) \leq 5$.
The condition Pisot is not necessary to have $L_\oplus(\beta)$ and $L_\otimes(\beta)$ are finite: L. S. Guimond, Z. Masáková and E. Pelantová in [8] prove that if β is an algebraic number such that at least one among its conjugates with modulus smaller than 1, so $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite.

In this paper, we define a similar concepts over the field of formal series. We begin in Section 1 by introducing the field of formal series and the β-expansion over this field. In Section 2, we will show that the condition Pisot is not necessary to have $L_\otimes(\beta)$ are finite. Especially, we give a sufficient condition for the conjugates of β to obtain $L_\odot(\beta) = 1$.
In Section 3, we prove the finiteness of $L_\odot(\beta)$ when β is Pisot or Salem formal power series and we give its exact value in the case of Pisot with some additional conditions.

2 Preliminaries

Let $\mathbb{F}_q$ be a finite field of q elements, $\mathbb{F}_q[x]$ is the ring of polynomials with coefficient in $\mathbb{F}_q$, $\mathbb{F}_q(x)$ is the field of rational functions and $\mathbb{F}_q(x,\beta)$ is the field of rational functions in base β. Let $\mathbb{F}_q((x^{-1}))$ be the field of formal power series of the form :

$$f = \sum_{k=-\infty}^{l} f_k x^k, \quad f_k \in \mathbb{F}_q$$

where

$$l = degf = \begin{cases} \max\{k : f_k \neq 0\} & \text{if } f \neq 0; \\ -\infty & \text{if } f = 0. \end{cases}$$

Define the absolute value by

$$|f| = \begin{cases} q^{deg\ f} & \text{if } f \neq 0; \\ 0 & \text{if } f = 0. \end{cases}$$

Since $|.|$ is not archimedean, $|.|$ fulfills the strict triangle inequality

$$|f+g| \leq \max\ (|f|, |g|) \ \text{ and } \ |f+g| = \max\ (|f|, |g|) \ \ if\ |f| \neq |g|.$$

Let $f \in \mathbb{F}_q((x^{-1}))$, define the integer (polynomial) part $[f] = \sum_{k=0}^{l} f_k x^k$ where the empty sum, as usual, is defined to be zero. So $[f] \in \mathbb{F}_q[x]$ and $f - [f] = \{f\} \in D(0,1)$.
An element $\beta \in \mathbb{F}_q((x^{-1}))$ is called algebraic integer of degree d if it verify $\beta^d + A_{d-1}\beta^{d-1} + \cdots + A_1\beta + A_0 = 0$ and is called unit series if $A_0 \in \mathbb{F}_q^*$.
β is Pisot (resp Salem) element if it is an algebraic integer over $\mathbb{F}_q[x]$, $|\beta| > 1$ and $|\beta_j| < 1$ for all Galois conjugates β_j (resp $|\beta_j| \leq 1$ and there exist at least one conjugate β_k such that $|\beta_k| = 1$).
In [3], Bateman and Duquette had characterized the Pisot and Salem element in $\mathbb{F}_q((x^{-1}))$.

Theorem 2.1 [3] *Let $\beta \in \mathbb{F}_q((x^{-1}))$ be an algebraic integer over $\mathbb{F}_q[x]$ and*

$$P(y) = y^n - A_{n-1}y^{n-1} - \cdots - A_0, \quad A_i \in \mathbb{F}_q[x],$$

be its minimal polynomial. Then
(i) β is a Pisot element if and only if $|A_{n-1}| > \max_{0 \leq j \leq n-2} |A_i|$.
(ii) β is a Salem element if and only if $|A_{n-1}| = \max_{0 \leq j \leq n-2} |A_i|$.

Let $\beta \in \mathbb{F}_q((x^{-1}))$ with $|\beta| > 1$. A representation in base β (or β-representation) of a formal series $f \in D(0,1)$ is an infinite sequence $(x_i)_{i \geq 1}$, $x_i \in \mathbb{F}_q[x]$, such that

$$f \ = \ \sum_{i \geq 1} \frac{x_i}{\beta^i}.$$

A particular β-representation of f is called the β-expansion of f in base β, noted $d_\beta(f)$, which is obtained by using the β-transformation T_β in the unit disk which is given by $T_\beta(f) = \beta f - [\beta f]$. Then $d_\beta(f) = (a_i)_{i \geq 1}$ where $a_i = [\beta T_\beta^{i-1}(f)]$, for better characterization of β-expansion, in [9], M.Hbaib and M.Mkaouar showed the following theorem.

Theorem 2.2 *[9] An infinite sequence $(a_i)_{i \geq 1}$ is the β-expansion of $f \in D(0,1)$ if and only if $|a_i| < |\beta|$ for all $i \geq 1$.*

Now let $f \in \mathbb{F}_q((x^{-1}))$ be an element with $|f| \geq 1$. Then there is a unique $k \in \mathbb{N}^*$ such that $|\beta|^{k-1} \leq |f| < |\beta|^k$, so $|\dfrac{f}{\beta^k}| < 1$ and we can represent f by shifting $d_\beta(\dfrac{f}{\beta^k})$ by k digits to the left. Therefore, if $d_\beta(f) = 0.a_1a_2a_3...$ then $d_\beta(\beta f) = a_1.a_2a_3....$
Let $Fin(\beta)$ be the set of f in $\mathbb{F}_q((x^{-1}))$ which have a finite β-expansion, so the β-expansion of every $f \in Fin(\beta)$ has this form:

$$d_\beta(f) = a_k a_{k-1} \cdots a_1 a_0 . a_{-1} a_{-2} \cdots a_m, \text{ where } m \in \mathbf{Z}.$$

The part $a_k a_{k-1} \cdots a_1 a_0$ is called the β-polynomial part of f and the part $a_{-1}a_{-2} \cdots a_m$ is called the β-fractional part of f.
We define also $deg_\beta(f) = k$ and $ord_\beta(f) = m$.
If $ord_\beta(f) \geq 0$ then f is called β-polynomial and the set of β-polynomials is denoted by $(\mathbb{F}_q[x])_\beta$, who is the analogue of $\mathbf{Z}_\beta$ in the real case.

We can define by an analogy with the real case the quantity $L_\oplus(\beta)$ and $L_\odot(\beta)$ as follows:

$$L_\oplus(\beta) = min\{n \in \mathbb{N} \setminus \forall f, g \in (\mathbb{F}_q[x])_\beta \text{ such that } f+g \in Fin(\beta) \Rightarrow \beta^n(f+g) \in (\mathbb{F}_q[x])_\beta\}$$

$$L_\odot(\beta) = min\{n \in \mathbb{N} \setminus \forall f, g \in (\mathbb{F}_q[x])_\beta \text{ such that } fg \in Fin(\beta) \Rightarrow \beta^n(fg) \in (\mathbb{F}_q[x])_\beta\}.$$

Minimum of an empty set is defined to be $+\infty$.

Remark 2.3 *Let us note that in the case of formal series the quantity $L_\oplus(\beta)$ is not interesting, because we know that in contrast to the real case, if f, $g \in \mathbb{F}_q((x^{-1}))$, we have $d_\beta(f+g) = d_\beta(f)+d_\beta(g)$, so the sum of two β-polynomials is always a β-polynomial.*

To calculate $L_\odot(\beta)$ for the families of basis β, we excluded the case when $deg(\beta) = 1$, since in this trivial case, the product of two β-polynomials is a β-polynomial, so we have $L_\odot(\beta) = 0$.

Let $\beta \in \mathbb{F}_q((x^{-1}))$ with $|\beta| > 1$ be an algebraic on $\mathbb{F}_q[x]$ and let $\beta_2, \ldots, \beta_n$ be the conjugates of β. We associate for every $f = \sum\limits_{i=0}^{n} a_i \beta^i$, the j-th conjugate which is defined by $f_j = \sum\limits_{i=0}^{n} a_i \beta_j^i$.

3 Main results

3.1 Sufficient conditions for finiteness of $L_\odot(\beta)$

In the following theorem, we will prove that $L_\odot(\beta)$ is finite when β is an algebraic integer. This result does not have an analogue to the real case.

Theorem 3.1 *Let $\beta \in \mathbb{F}_q((x^{-1}))$, with $\mid \beta \mid > 1$ be an algebraic integer on $\mathbb{F}_q[x]$. Then $L_\odot(\beta)$ is finite.*

Proof Let $f = a_s\beta^s + a_{s-1}\beta^{s-1} + ... + a_0$ and $g = b_m\beta^m + b_{m-1}\beta^{m-1} + ... + b_0$, where $\mid a_i \mid<\mid \beta \mid$ for all $0 \leq i \leq s$ and $\mid b_j \mid<\mid \beta \mid$ for all $0 \leq j \leq m$, such that $fg \in Fin(\beta)$. Therefore, we can write

$fg = c_n\beta^n+c_{n-1}\beta^{n-1}+...+c_0+c_{-1}\beta^{-1}+c_{-2}\beta^{-2}+...+c_{-k}\beta^{-k}$, with $\mid c_j \mid<\mid \beta \mid$ for all $-k \leq j \leq n$.

we denote by $h_j = c_{-1}\beta_j^{-1} + c_{-2}\beta_j^{-2} + ... + c_{-k}\beta_j^{-k}$ for all $1 \leq j \leq d$, where $\beta_1 = \beta$ and β_j, $2 \leq j \leq d$ are the Galois conjugates of β and d is the algebraic degree of β. Then

$$h_j = \sum_{i=0}^{d-1} \alpha_i\beta_j^{-i} \text{ where } \alpha_i \in \mathbb{F}_q[x] \text{ for all } 0 \leq i \leq d-1.$$

We have $\mid h \mid=\mid h_1 \mid< 1$, if $\mid \beta_j \mid\geq 1$ then $\mid h_j \mid=\mid c_{-1}\beta_j^{-1} + c_{-2}\beta_j^{-2} + ... + c_{-k}\beta_j^{-k} \mid\leq\mid \beta \mid$, and if $\mid \beta_j \mid< 1$ then $\mid h_j \mid=\mid (a_s\beta_j^s + a_{s-1}\beta_j^{s-1} + ... + a_0).(b_m\beta_j^m + b_{m-1}\beta_j^{m-1} + ... + b_0) - (c_n\beta_j^n + c_{n-1}\beta_j^{n-1} + ... + c_0) \mid\leq\mid \beta \mid^2$.
Since the matrix $M = (\beta_j^{-k})_{1\leq j\leq d;0\leq k\leq d-1}$ is non singular, we give that

$$\begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{d-1} \end{pmatrix} = M^{-1} \begin{pmatrix} h_1 \\ \vdots \\ h_d \end{pmatrix}.$$

This implies that $\mid \alpha_i \mid< C(\beta)$, where $C(\beta)$ is a constant depend only on β, therefore the number of elements $(\alpha_i)_{0\leq i\leq d-1}$ is finite. So $L_\odot(\beta)$ is finite. $\square$

3.2 Computation of $L_\odot(\beta)$

We propose the following quantitative study over this family of algebraic formal series β, that have at least one of its conjugates, say β_j, in absolute value smaller than $\dfrac{1}{\mid \beta \mid}$.

Theorem 3.2 *Let $\beta \in \mathbb{F}_q((x^{-1}))$ be an algebraic on $\mathbb{F}_q[x]$ with $deg(\beta) \geq 2$, which have a conjugate, say β_j verifying $|\beta_j| \leq \dfrac{1}{\mid \beta \mid}$. Then $L_\odot(\beta) \in \{0, 1\}$.*

Proof.
Let $f = \sum\limits_{i=0}^{s} a_i\beta^i$ and $g = \sum\limits_{i=0}^{r} b_i\beta^i$ where $|a_i| < |\beta|$ and $|b_i| < |\beta|$, such that $fg \in Fin(\beta)$.
Let $f_j = \sum\limits_{i=0}^{s} a_i\beta_j^i$ and $g_j = \sum\limits_{i=0}^{r} b_i\beta_j^i$. Since $|\beta_j| \leq \dfrac{1}{\mid \beta \mid}$, we have $|f_j| < |\beta|$ and $|g_j| < |\beta|$.
So

$$|f_jg_j| < |\beta|^2$$

We assume that $d_\beta(fg) = c_h...c_0.c_{-1}...c_{-m}$ where $c_{-m} \neq 0$, so we have

$$|f_jg_j| = |\sum_{i=-m}^{h} c_i\beta_j^i| = |c_{-m}\beta_j^{-m}| \geq |\beta|^m.$$

Therefore $m < 2$. $\square$
As application of the above theorem, we have treat some specific cases.

Corollary 3.3 *Let β be a quadratic Pisot unit with deg$(\beta) \geq 2$. Then $L_{\odot}(\beta) = 1$.*

Proof.
In this case β verify $\beta^2 + A_1\beta + A_0 = 0$ where $A_0 \in \mathbb{F}_q^*$, so the unique conjugate of β is β_j such that $|\beta_j| = \dfrac{1}{\mid \beta \mid}$.
By Theorem 3.2, we obtain $L_{\odot}(\beta) \in \{0, 1\}$. Let now $A_1 = c_d x^d + ... + c_0$ where $d = deg(\beta) \geq 2$ and $c_d \neq 0$, we have $x, c_d.x^{d-1} \in (\mathbb{F}_q[x])_\beta$ and
$c_d x^d = -\beta - (c_{d-1}x^{d-1} + ... + c_0) - \frac{A_0}{\beta}$. So, $L_{\odot}(\beta) = 1$. □

Corollary 3.4 *Let β be a cubic Salem unit with deg$(\beta) \geq 2$. Then $L_{\odot}(\beta) \in \{0, 1\}$.*

Proof.
In this case β verify $\beta^3 + A_2\beta^2 + A_1\beta + A_0 = 0$, where $|A_1| = |A_2| = |\beta|$ and $A_0 \in \mathbb{F}_q^*$.
So β have two conjugates β_2 and β_3 such that $|\beta_2| = \dfrac{1}{|\beta|}$ et $|\beta_3| = 1$.
By Theorem 3.2, we obtain $L_{\odot}(\beta) \in \{0, 1\}$. □

Corollary 3.5 *Let β be a Salem unit with deg$(\beta) \geq 2$ verifying:*

$$\beta^d + A_{d-1}\beta^d + \ldots + A_1\beta + A_0 = 0, \tag{3.1}$$

where $A_0 \in \mathbb{F}_q^*$ and $\mid A_1 \mid = \mid A_{d-1} \mid$. Then $L_{\odot}(\beta) \in \{0, 1\}$.

Proof.
Let $\beta = \beta_1$ be a Salem unit verifying (3.1). Then

$$\mid A_0 \mid = \mid \prod_{1 \leq i \leq d} \beta_i \mid = 1 \text{ where } \beta_2, \ldots, \beta_d \text{ its conjugates}$$

$$\mid A_{d-1} \mid = \mid \sum_{i=1}^{d} \beta_i \mid = \mid A_1 \mid = \mid \sum_{1 \leq i_1 < \ldots < i_{d-1} \leq d} \beta_{i_1} \ldots \beta_{i_{d-1}} \mid$$

If there exist β_i and β_j $(i \neq j)$ such that $|\beta_i| < 1$ and $|\beta_j| < 1$, then we obtain in this case $\mid A_1 \mid < \mid \beta \mid$ which contradicts the hypothesis that

$$\mid \beta \mid = \mid A_{d-1} \mid = \mid A_1 \mid .$$

we conclude that β have a unique conjugate β_j such that $\mid \beta_j \mid < 1$ and the other conjugates of equal absolute value 1. So $\mid \beta_j \mid = \dfrac{1}{\mid \beta \mid}$ and by Theorem 3.2, we obtain $L_{\odot}(\beta) \in \{0, 1\}$. □
The following theorem, allows us to calculate $L_{\odot}(\beta)$ for some Pisot series.

Theorem 3.6 *let β be a Pisot series satisfying*

$$\beta^d + A_{d-1}\beta^{d-1} + \cdots + A_0 = 0$$

such that deg(β) $= m \geq 2$ and deg(A_0) $= s \geq$ deg(A_i), for all $0 \leq i \leq d-2$. Then for any $n \geq m$ the beta-expansion of x^n is given by:

$$d_\beta(x^n) = a^n_{\lambda(n)} \cdots a^n_0.a^n_{-1} \cdots a^n_{-\mu(n)}.$$

Where λ and μ are two increasing sequences defined by:

$$\mu(n) = -ord_\beta(x^n) = (d-1)[\frac{n-s}{m-s}] \text{ and } deg(a^n_{-\mu(n)}) = s + (\overline{n-s}) = \sup_{0 \leq k \leq d-2} deg(a^n_{-\mu(n)+k})$$

where $(\overline{n-s})$ is the rest of the Euclidean division of $(n-s)$ by $(m-s)$.

To prove the above theorem, we will need the following lemma:

Lemma 3.7 *Let β be a Pisot series. Then $ord_\beta(x^{n+1}) \leq ord_\beta(x^n)$ for all $n \in \mathbb{N}^*$.*

Proof.
Let $P(y) = y^d + A_{d-1}y^{d-1} + \cdots + A_0$ be the minimal polynomial of β and c be the dominant coefficient of A_{d-1}, Let $m =$deg(β), for $n < m-1$ we have $ord_\beta(x^{n+1}) = ord_\beta(x^n) = 0$. Let now $n \geq m$ we suppose that

$$d_\beta(x^n) = a^n_{\lambda(n)} \cdots a^n_0.a^n_{-1} \cdots a^n_{-\mu(n)} \text{ with } a^n_{-\mu(n)} \neq 0.$$

We will show by induction on n that $\mu(n+1) \geq \mu(n)$ and $\deg(a^n_{-\mu(n)}) \geq s$, where $s =\deg A_0$.
So we have

$$x^m = -c^{-1}\beta - c^{-1}(A_{d-1} - cx^m) - \cdots - c^{-1}A_0\beta^{-(d-1)}.$$

this implies

$$d_\beta(x^m) = a^m_{\lambda(m)} a^m_0.a^m_{-1} \cdots a^m_{-\mu(m)}$$

$$\text{where } \begin{cases} a^m_{\lambda(m)} = -c^{-1} \\ \vdots \\ a^m_{-\mu(m)} = -c^{-1}A_0 \end{cases} \text{ and } \lambda(m) = 1, \ \mu(m) = d-1$$

Let now

$$\mathcal{A}_n = \{0 \leq i \leq d-1 \text{ such that } \deg(a^n_{-\mu(n)+i}) = m-1\}$$

If $\mathcal{A}_n = \emptyset$, then $a^{n+1}_{-\mu(n+1)} = xa^n_{-\mu(n)}$ and $\mu(n+1) = \mu(n)$ and $\deg(a^n_{-\mu(n+1)}) = 1 + \deg(a^n_{-\mu(n)})$.
If $\mathcal{A}_n \neq \emptyset$, taking $k_n = min\mathcal{A}_n$ and γ be the dominant coefficient of $a^n_{-\mu(n)+k_n}$.

- If $k_n = d-1$, then $a^{n+1}_{-\mu(n+1)} = -c^{-1}.\gamma.A_0 + x.a^n_{-\mu(n)}$ and $\mu(n+1) = \mu(n)$.
- If $k_n < d-1$, then $a^{n+1}_{-\mu(n+1)} = -c^{-1}\gamma A_0$ and $\mu(n+1) = \mu(n) + d - 1 - k_n$.

Lemma 3.8 *Let β be a Pisot series, such that $m =$deg(β) ≥ 2.
Then $L_\odot(\beta) = -ord_\beta(x^{2m-2})$.*

Proof.
Let $f = a_n\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_0$ and $g = b_m\beta^m + b_{m-1}\beta^{m-1} + \cdots + b_0$ in $\mathbb{F}_q[x]_\beta$ such that $f.g \in \mathrm{Fin}(\beta)$, since

$$f.g = \sum_{k=0}^{n+m} (\sum_{p=0}^{k} b_p a_{k-p})\beta^k.$$

We have $-ord_\beta(f.g) \leq max\{-ord_\beta(b_p a_{k-p}); 0 \leq p \leq k \leq n+m\}$, using the above lemma we get $L_\odot(\beta) \leq -ord_\beta(x^{2m-2})$. Or $f = x^{m-1} \in \mathbb{F}_q[x]_\beta$ and $-ord_\beta(f^2) = -ord_\beta(x^{2m-2})$, so $L_\odot(\beta) = -ord_\beta(x^{2m-2})$. $\square$

Proof of Theorem 3.6.
We will show the result by induction on $n \geq m$.
For $n = m$, we have

$$x^m = -c^{-1}\beta - (A_{d-1} - cx^m) - \cdots - c^{-1}A_0\beta^{-(d-1)}$$

where c is the dominant coefficient of A_{d-1}, so

$$-ord_\beta(x^m) = d - 1 = \mu(m) \text{ and } deg(a^m_{-\mu(m)}) = deg(-c^{-1}A_0) = s = s + (\overline{m-s}).$$

So the result is true for $n = m$.
Assume that $\mu(n) = -ord_\beta(x^n) = (d-1)[\frac{n-s}{m-s}]$ and $\deg(a^n_{-\mu(n)}) = s + (\overline{n-s})$. We have

$$x^n = a^n_{\lambda(n)}\beta^{\lambda(n)} + \cdots + a^n_0 + a^n_{-1}\beta^{-1} + \cdots + a^n_{-\mu(n)}\beta^{\mu(n)}.$$

Therefore

$$x^{n+1} = xa^n_{\lambda(n)}\beta^{\lambda(n)} + \cdots + xa^n_0 + xa^n_{-1}\beta^{-1} + \cdots + xa^n_{-\mu(n)}\beta^{\mu(n)}.$$

We distinguish two cases:
Case 1: $\deg(a^n_{-\mu(n)}) = m - 1$, in this case we have $(\overline{n-s}) = m - s - 1$, so $(n-s) = [\frac{n-s}{m-s}](m-s) + m - s - 1$, this implies $(n+1-s) = ([\frac{n-s}{m-s}] + 1)(m-s)$, hence $[\frac{n+1-s}{m-s}] = [\frac{n-s}{m-s}] + 1$ and $(\overline{n+1-s}) = 0$. Also we have

$$\begin{aligned}
\mu(n+1) &= -ord_\beta(x^{n+1}) = -ord_\beta(x^n) + d - 1 \\
&= (d-1)[\frac{n-s}{m-s}] + (d-1) \\
&= (d-1)([\frac{n-s}{m-s}] + 1) \\
&= (d-1)[\frac{n+1-s}{m-s}]
\end{aligned}$$

and $a^{n+1}_{-\mu(n+1)} = -c^{-1}.\gamma.A_0$ where γ is the dominant coefficient of $a^n_{-\mu(n)}$, so $\deg(a^{n+1}_{-\mu(n+1)}) = s = s + (\overline{n+1-s})$
Case 2: $\deg(a^n_{-\mu(n)}) < m - 1$, in this case we have

$$\mu(n+1) = -ord_\beta(x^{n+1}) = -ord_\beta(x^n) = \mu(n)$$

and

$$a^{n+1}_{-\mu(n+1)} = \begin{cases} xa^n_{-\mu(n)} & \text{if } \deg(a^n_{-\mu(n)+d-1}) < m-1, \\ -c^{-1}.\delta.A_0 + xa^n_{-\mu(n)} & \text{if } \deg(a^n_{-\mu(n)+d-1}) = m-1. \end{cases}$$

where δ is the dominant coefficient of $a^n_{-\mu(n)+d-1}$.
Therefore $\deg(a^{n+1}_{-\mu(n+1)}) = 1+ \deg(a^n_{-\mu(n)}) = 1 + s + (\overline{n-s}) = s + (\overline{n+1-s})$. $\square$

Corollary 3.9 *let β be a Pisot series satisfying*

$$\beta^d + A_{d-1}\beta^{d-1} + \cdots + A_0 = 0$$

such that $deg(\beta) = m \geq 2$ and $deg(A_0) = s \geq deg(A_i)$ $\forall 0 \leq i \leq d-2$. Then $L_\odot(\beta) = (d-1)([\frac{m-2}{m-s}]+1)$.

Corollary 3.10 *Let β be a Pisot series verifying $\beta^2+A_1\beta+A_0 = 0$ such that $deg(\beta) = m$ and $deg(A_0) = s$. Then*

$$L_\odot(\beta) = [\frac{m-2}{m-s}] + 1$$

4 Open Problem

Study other finiteness conditions and explicitly calculate $L_\odot(\beta)$ for other families of algebraic formal power series.
Is it possible to find conditions on β in order that $L_\odot(\beta) = 0$, meaning the set $(\mathbb{F}_q[x])_\beta$ is stable under multiplication?

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