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Quasi-static transmission problem in thermo-viscoplasticity

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Abstract

We consider a mathematical model which describes the quasi-static transmission problem in thermo-viscoplasticity. We derive a weak formulation of the system of motion equation and energy equation. We prove the existence and uniqueness of the solution and some properties of the solution.

Keywords: *Transmission, thermo-viscoplastic, temperature, interface.*
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1 Introduction

The thermo-viscoplastic constitutive laws has been studied by mathematicians, physicists and engineers in order to model the effect of temperature in the behaviour of some real bodies like metals, magmas, polymers and so on, see for examples and details [6], [12] and [13]. Examples and mechanical interpretation of viscoplasticity and thermo-viscoplasticity can be found in [3], [5], [7] and [16].

The aim of this paper is to study the quasi-static transmission problem between two thermo-viscoplastic bodies. For this, we consider a rate-type constitutive equation of the form

$$\frac{\partial \boldsymbol{\sigma}}{\partial t}=\mathcal{A}\left(\varepsilon\left(\frac{\partial \mathbf{u}}{\partial t}\right)\right)+\mathcal{G}\left(\boldsymbol{\sigma},\varepsilon\left(\mathbf{u}\right),\theta\right),$$

in which $\mathbf{u}$, $\boldsymbol{\sigma}$ represent, respectively, the displacement field and the stress field, θ represents the absolute temperature, $\mathcal{A}$ is a real tensor describing the purely viscous property of the material and $\mathcal{G}$ is a nonlinear constitutive function which describes the thermo-plastic behaviour of the material. Situations of such problem are very common in industry, geology and everyday life such as superposition of tectonic plates and development of multi-layer bodies like composite materials.

Transmission problems in mechanics of continuum media, in particular for elastic and thermoelastic models, are the topic of numerous papers [1], [11] and [15].

2 Preliminaries

We consider a mathematical problem modelling the quasi-static transmission problem between two thermo-viscoplastic bodies. To this end, let us consider a bounded domain $\Omega\subset\mathbb{R}^n$ $(n=2,3)$ with a Lipschitz boundary $\partial\Omega$. The domain is partitioned into two parts Ω_1 and Ω_2 separated with a Lipschitz hypersurface Γ_0. We denote by Γ_1 and Γ_2 the boundaries

$$\Gamma_1=\partial\Omega_1\cap\partial\Omega \text{ and } \Gamma_2=\partial\Omega_2\cap\partial\Omega,$$

and we suppose that Γ_1 and Γ_2 are measurable domains with $meas\left(\Gamma_2\right)>0$.

We denote by $\mathbb{S}_n$ the space of symmetric tensors on $\mathbb{R}^n$. We define the inner product and the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{S}_n$, respectively, by

$$\mathbf{u}\cdot\mathbf{v}=u_iv_i \quad \forall\ \mathbf{u},\ \mathbf{v}\in\mathbb{R}^n \text{ and } \boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij} \quad \forall\ \boldsymbol{\sigma},\ \boldsymbol{\tau}\in\mathbb{S}_n,$$
$$\left|\mathbf{u}\right|=\left(\mathbf{u}\cdot\mathbf{u}\right)^{\frac{1}{2}} \quad \forall\mathbf{u}\in\mathbb{R}^n \text{ and } \left|\boldsymbol{\sigma}\right|=\left(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\right)^{\frac{1}{2}} \quad \forall\boldsymbol{\sigma}\in\mathbb{S}_n.$$

Here and below, the indices i and j run from 1 to n and Einstein's convention is used.

For $p=1,2$ we shall use the notations

$$L^2\left(\Omega_p\right)_s^{n\times n}=\left\{\boldsymbol{\sigma}=\left\{\sigma_{ij}\right\}\in L^2\left(\Omega_p\right)^{n\times n}\mid\sigma_{ij}=\sigma_{ji}\right\},$$
$$H_p=\left\{\mathbf{u}=\left\{\sigma_i\right\}\in L^2\left(\Omega_p\right)^n\mid\varepsilon\left(\mathbf{u}\right)\in L^2\left(\Omega_p\right)_s^{n\times n}\right\},$$
$$\mathcal{H}_p=\left\{\boldsymbol{\sigma}\in L^2\left(\Omega_p\right)_s^{n\times n}\mid Div\left(\boldsymbol{\sigma}\right)\in L^2\left(\Omega_p\right)^n\right\},$$
$$\mathcal{V}=\left\{\left(\mathbf{v}_1,\mathbf{v}_2\right)\in H_1\times H_2\mid\mathbf{v}_1-\mathbf{v}_2=0\text{ on }\Gamma_0\text{ and }\mathbf{v}_2=0\text{ on }\Gamma_2\right\},$$
$$\mathcal{W}=\left\{\left(\zeta_1,\zeta_2\right)\in H^1\left(\Omega_1\right)\times H^1\left(\Omega_2\right)\mid\zeta_1-\zeta_2=0\text{ on }\Gamma_0\right\}.$$

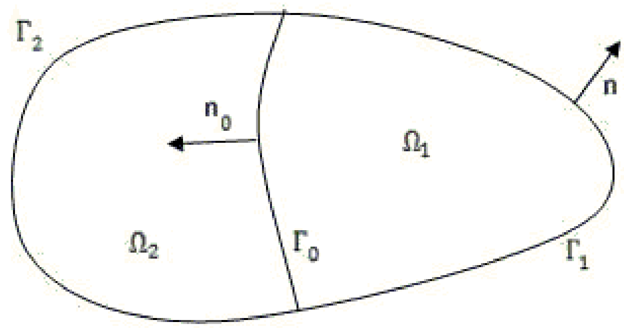

Figure 1: Transmission Domain

Here $\varepsilon : H_p \longrightarrow L^2(\Omega_p)_s^{n\times n}$ and $Div : \mathcal{H}_p \longrightarrow L^2(\Omega_p)^n$ are, respectively, the deformation and the divergence operators defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{and} \quad Div(\boldsymbol{\sigma}) = (\sigma_{ij,j}).$$

The spaces H_p, $\mathcal{H}_p$, $\mathcal{V}$ and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{gathered}
(\mathbf{u}, \mathbf{v})_{H_p} = (\mathbf{u}, \mathbf{v})_{L^2(\Omega_p)^n} + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{L^2(\Omega_p)_s^{n\times n}}, \\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_p} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{L^2(\Omega_p)_s^{n\times n}} + (Div(\boldsymbol{\sigma}), Div(\boldsymbol{\tau}))_{L^2(\Omega_p)^n}, \\
((\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2))_{\mathcal{V}} = (\mathbf{u}_1, \mathbf{u}_2)_{H_1} + (\mathbf{v}_1, \mathbf{v}_2)_{H_2}, \\
((\zeta_1, \zeta_2), (\xi_1, \xi_2))_{\mathcal{W}} = (\zeta_1, \zeta_2)_{H^1(\Omega_1)} + (\xi_1, \xi_2)_{H^1(\Omega_2)}.
\end{gathered}$$

The associated norms on the spaces H_p, $\mathcal{H}_p$, $\mathcal{V}$ and $\mathcal{W}$ are denoted by $\|.\|_{H_p}$, $\|.\|_{\mathcal{H}_p}$, $\|.\|_{\mathcal{V}}$ and $\|.\|_{\mathcal{W}}$, respectively. Since the boundaries $\partial\Omega$ and Γ_0, respectively, are Lipschitz continuous, the unit outward normal vector fields $\mathbf{n}$ and $\mathbf{n}_0$ are defined a.e on $\partial\Omega$ and Γ_0, respectively, where $\mathbf{n}_0$ is supposed to be oriented to the exterior of Ω_1 and to the interior of Ω_2, see figure 1.

Moreover, since $meas(\Gamma_2) > 0$, Korn's inequality holds and thus, there exists a positive constant C_0 depending only on Ω, Γ_2 such that

$$\|(\varepsilon(\mathbf{v}_1), \varepsilon(\mathbf{v}_2))\|_{L^2(\Omega_1)_s^{n\times n} \times L^2(\Omega_2)_s^{n\times n}} \geq C_0 \|(\mathbf{v}_1, \mathbf{v}_2)\|_{H_1 \times H_2} \ \ \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V}.$$

Let introduce the space $H_{\Gamma_p} = \left(H^{\frac{1}{2}}(\Gamma_p \cup \Gamma_0)\right)^n$ $p = 1, 2$ and denoting by $\gamma : H_p \longrightarrow H_{\Gamma_p}$ the trace map.

If $\boldsymbol{\sigma} \in \mathcal{H}_p$ the Cauchy stress tensor $\boldsymbol{\sigma\nu}$ exists, such that $\boldsymbol{\sigma\nu} \in H'_{\Gamma_p}$ and for which the following Green formula holds

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{L^2(\Omega_p)_s^{n\times n}} + (Div(\boldsymbol{\sigma}), \mathbf{v})_{L^2(\Omega_p)^n} = (\boldsymbol{\sigma\nu}, \gamma\mathbf{v})_{H'_{\Gamma_p}\times H_{\Gamma_p}} \quad \forall \mathbf{v} \in H_p.$$

In addition, if $\boldsymbol{\sigma}$ is sufficiently regular (say $\mathcal{C}^1$), then

$$\begin{aligned}(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{L^2(\Omega_p)_s^{n\times n}} + (Div(\boldsymbol{\sigma}), \mathbf{v})_{L^2(\Omega_p)^n} &= \int_{\Gamma_p} \boldsymbol{\sigma}\cdot\mathbf{n}\cdot\mathbf{v}d\gamma \\ +(-1)^{p+1}\int_{\Gamma_0} \boldsymbol{\sigma}\cdot\mathbf{n}_0\cdot\mathbf{v}d\gamma_0 \; \forall\mathbf{v} &\in H_p, \quad p=1,2.\end{aligned}$$

where $d\gamma$ and $d\gamma_0$ denote, respectively, the surface element on Γ_p and Γ_0.

The physical setting is the following. Two thermo-viscoplastic bodies occupy, respectively, the domains Ω_1 and Ω_2. We assume that the second body is clamped on $\Gamma_2 \times (0,T)$, $(T>0)$ and therefore the displacement field vanishes there. Surface tractions of density $\mathbf{g}$ act on $\Gamma_1 \times (0,T)$. On the interface $\Gamma_0 \times (0,T)$ we impose transmission boundary conditions between the two bodies. Volume forces of density $\mathbf{f}_p$ is applied in $\Omega_p \times (0,T)$, $p=1,2$. In addition, we admit possible external heat source applied in $\Omega_p \times (0,T)$, given by the function r_p, $p=1,2$.

The mechanical problem may be formulated as follows.

Problem (P). For $p=1,2$, find the displacement field $\mathbf{u}_p : \Omega_p \times (0,T) \longrightarrow \mathbb{R}^n$, the stress field $\boldsymbol{\sigma}_p : \Omega_p \times (0,T) \longrightarrow \mathbb{S}_n$ and the temperature $\theta_p : \Omega_p \times (0,T) \longrightarrow \mathbb{R}$ such that

$$\frac{\partial\boldsymbol{\sigma}_p}{\partial t} = \mathcal{A}_p\left(\varepsilon\left(\frac{\partial\mathbf{u}_p}{\partial t}\right)\right) + \mathcal{G}_p(\boldsymbol{\sigma}_p, \varepsilon(\mathbf{u}_p), \theta_p) \text{ in } \Omega_p \times (0,T), \tag{2.1}$$

$$Div(\boldsymbol{\sigma}_p) + \mathbf{f}_p = 0 \text{ in } \Omega_p \times (0,T), \tag{2.2}$$

$$\frac{\partial\theta_p}{\partial t} - \operatorname{div}(k_p\nabla\theta_p) = \varphi_p(\boldsymbol{\sigma}_p, \varepsilon(\mathbf{u}_p(t)), \theta_p) + r_p \text{ in } \Omega_p \times (0,T), \tag{2.3}$$

$$\mathbf{u}_1 - \mathbf{u}_2 = 0 \text{ on } \Gamma_0 \times (0,T), \tag{2.4}$$

$$\boldsymbol{\sigma}_1\cdot\mathbf{n}_0 - \boldsymbol{\sigma}_2\cdot\mathbf{n}_0 = 0 \text{ on } \Gamma_0 \times (0,T), \tag{2.5}$$

$$\theta_1 - \theta_2 = 0 \text{ on } \Gamma_0 \times (0,T), \tag{2.6}$$

$$k_1\frac{\partial\theta_1}{\partial\mathbf{n}_0} - k_2\frac{\partial\theta_2}{\partial\mathbf{n}_0} = 0 \text{ on } \Gamma_0 \times (0,T), \tag{2.7}$$

$$\boldsymbol{\sigma}_1\cdot\mathbf{n} = \mathbf{g} \text{ on } \Gamma_1 \times (0,T), \tag{2.8}$$

$$\mathbf{u}_2 = 0 \text{ on } \Gamma_2 \times (0,T), \tag{2.9}$$

$$\frac{\partial\theta_1}{\partial\mathbf{n}} = 0 \text{ on } \Gamma_1 \times (0,T), \tag{2.10}$$

$$\frac{\partial \theta_2}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_2 \times (0,T)\,, \tag{2.11}$$

$$\mathbf{u}_p\,(0) = \mathbf{u}_{p0},\ \boldsymbol{\sigma}_p\,(0) = \boldsymbol{\sigma}_{p0} \text{ and } \theta_p\,(0) = \theta_{p0} \text{ in } \Omega_p. \tag{2.12}$$

This problem represents a quasi-static transmission problem between two thermo-viscoplastic bodies. Equation (2.1) is the thermo-viscoplastic constitutive law where $\mathcal{A}_p$ is a real tensor describing the purely viscous property of the material and $\mathcal{G}_p$ is a nonlinear constitutive function which describes the plastic behaviour of the material. (2.2) represents the quasi-static equilibrium equation. Equation (2.3) represents the energy conservation where φ_p is a nonlinear constitutive function which describes the heat generated by the work of internal forces.

(2.4), (2.5), (2.6) and (2.7) are the transmission conditions on the interface $\Gamma_0 \times (0,T)$ for the displacement field, the stress field and the temperature. Equalities (2.8) and (2.9) are the displacement-traction boundary conditions, respectively. (2.10) and (2.11) represent homogeneous Neumann boundary conditions for the temperatures. Finally the functions $\mathbf{u}_{p0}$, $\boldsymbol{\sigma}_{p0}$ and θ_{p0} in (2.12) represent the initial data.

In the study of the mechanical problem (P) we consider, for $p = 1, 2$, the following hypotheses :

$$\left\{\begin{array}{l}
\mathcal{A}_p : \Omega_p \times \mathbb{S}_n \longrightarrow \mathbb{S}_n \text{ is a symmetric and positively definite bounded} \\
\text{tensor, i.e.:} \\
\text{(a) } \mathcal{A}_{p_{ijkh}} \in L^\infty(\Omega_p) \quad \forall i\ j,k,h = \overline{1,n}. \\
\text{(b) } \mathcal{A}_p\,(x)\,\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathcal{A}_p\,(x)\,\boldsymbol{\tau} \quad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{S}_n,\ \text{a.e. in } \Omega_p. \\
\text{(c) There exists an } \alpha > 0 \text{ such that} \\
\mathcal{A}_p\,(x)\,\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} \geq \alpha_{\mathcal{A}_p}\,|\boldsymbol{\sigma}|^2\ \forall \mathbf{v} \in \mathbb{S}_n, \quad \text{a.e. in } \Omega_p.
\end{array}\right. \tag{2.13}$$

$$\left\{\begin{array}{l}
\mathcal{G}_p : \Omega_p \times \mathbb{S}_n \times \mathbb{S}_n \times \mathbb{R} \longrightarrow \mathbb{S}_n \text{ has the following properties:} \\
\text{(a) There exists an } L_{\mathcal{G}_p} > 0 \text{ such that} \\
|\mathcal{G}_p\,(x,\boldsymbol{\sigma}_1,\boldsymbol{\tau}_1,\xi_1) - \mathcal{G}_p\,(x,\boldsymbol{\sigma}_2,\boldsymbol{\tau}_2,\xi_2)| \leq \\
\qquad\qquad L_{\mathcal{G}_p}\,(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2| + |\xi_1 - \xi_2|) \\
\forall \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\tau}_1,\boldsymbol{\tau}_2 \in S_n,\ \forall \xi_1,\xi_2 \in \mathbb{R} \text{ a.e. } x \in \Omega_p. \\
\text{(b) The mapping } x \longrightarrow \mathcal{G}_p\,(x,\boldsymbol{\sigma},\boldsymbol{\tau},\xi) \text{ is Lebesgue measurable} \\
\text{on } \Omega_p\ \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{S}_n,\ \forall \xi \in \mathbb{R}. \\
\text{(c) The mapping } x \longrightarrow \mathcal{G}_p\,(x,\mathbf{0},\mathbf{0},0) \in L^2\,(\Omega_p)_s^{n\times n}.
\end{array}\right. \tag{2.14}$$

$$\left\{\begin{array}{l}\varphi_p : \Omega \times \mathbb{S}_n \times \mathbb{S}_n \times \mathbb{R} \longrightarrow \mathbb{R} \text{ has the following properties :}\\ \text{(a) There exists an } L_{\varphi_p} > 0 \text{ such that}\\ \left|\varphi_p(x, \boldsymbol{\sigma}_1, \boldsymbol{\tau}_1, \xi_1) - \varphi_p(x, \boldsymbol{\sigma}_2, \boldsymbol{\tau}_2, \xi_2)\right| \leq \\ \qquad\qquad L_{\varphi_p}(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2| + |\xi_1 - \xi_2|)\\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{S}_n,\ \forall \xi_1, \xi_2 \in \mathbb{R} \text{ a.e. } x \in \Omega_p.\\ \text{(b) The mapping } x \longrightarrow \varphi_p(x, \boldsymbol{\sigma}, \boldsymbol{\tau}, \xi) \text{ is Lebesgue measurable}\\ \text{on } \Omega_p\ \ \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}_n,\ \forall \xi \in \mathbb{R}.\\ \text{(c) The mapping } x \longrightarrow \varphi_p(x, \mathbf{0}, \mathbf{0}, 0) \in L^2(\Omega_p).\end{array}\right. \tag{2.15}$$

$$\left\{\begin{array}{l}\mathbf{f}_p \in W^{1,\infty}(0, T; L^2(\Omega_p)^n),\ \mathbf{g} \in W^{1,\infty}(0, T; L^2(\Gamma_1)),\\ \qquad\qquad r_p \in L^2(0, T; L^2(\Omega_p)).\end{array}\right. \tag{2.16}$$

$$(\mathbf{u}_{10}, \mathbf{u}_{20}) \in \mathcal{V},\ (\boldsymbol{\sigma}_{10}, \boldsymbol{\sigma}_{20}) \in \mathcal{H}_1 \times \mathcal{H}_1,\ (\theta_{10}, \theta_{20}) \in \mathcal{W}. \tag{2.17}$$

$$k_p \in L^\infty(\Omega_p),\ k_p(x) \geq k_p^* > 0 \text{ a.e. on } \Omega_p. \tag{2.18}$$

We also admits the hypothesis of continuity

$$(\boldsymbol{\sigma}_{10}, \varepsilon(\mathbf{u}_{10}))_{L^2(\Omega_1)_s^{n\times n}} + (\boldsymbol{\sigma}_{20}, \varepsilon(\mathbf{u}_{20}))_{L^2(\Omega_2)_s^{n\times n}} = (\mathbf{f}_1(0), \mathbf{u}_{10})_{L^2(\Omega_1)^n} + (\mathbf{f}_2(0), \mathbf{u}_{20})_{L^2(\Omega_2)^n} + \int_{\Gamma_1} \mathbf{g}(0) \cdot \mathbf{u}_{10} d\gamma. \tag{2.19}$$

Moreover, we remark that hypothesis (2.13) implies the existence of a positive constant $m_{\mathcal{A}}$ such that

$$\|\mathcal{A}_p \boldsymbol{\sigma}\|_{L^2(\Omega_p)_s^{n\times n}} \leq m_{\mathcal{A}_p} \|\boldsymbol{\sigma}\|_{L^2(\Omega_p)_s^{n\times n}} \quad \forall \boldsymbol{\sigma} \in L^2(\Omega_p)_s^{n\times n}. \tag{2.20}$$

Using the above notations and Green's formula, we can easily derive the following variational formulation of the mechanical problem (P).

Problem (PV). For $p = 1, 2$, find the displacement field $\mathbf{u}_p : \Omega_p \times (0, T) \longrightarrow \mathbb{R}^n$, the stress field $\boldsymbol{\sigma}_p : \Omega_p \times (0, T) \longrightarrow \mathbb{S}_n$ and the temperature $\theta_p : \Omega_p \times (0, T) \longrightarrow \mathbb{R}$ such that

$$\frac{\partial \boldsymbol{\sigma}_p}{\partial t} = \mathcal{A}_p\left(\varepsilon\left(\frac{\partial \mathbf{u}_p}{\partial t}\right)\right) + \mathcal{G}_p(\boldsymbol{\sigma}_p(t), \varepsilon(\mathbf{u}_p(t)), \theta_p(t)) \text{ a.e. } t \in (0, T), \tag{2.21}$$

$$(\boldsymbol{\sigma}_1(t), \varepsilon(\mathbf{v}_1))_{L^2(\Omega_1)_s^{n\times n}} + (\boldsymbol{\sigma}_2(t), \varepsilon(\mathbf{v}_2))_{L^2(\Omega_2)_s^{n\times n}} = \int_{\Gamma_1} \mathbf{g}(t) \cdot \mathbf{v}_1 d\gamma + (\mathbf{f}_1(t), \mathbf{v}_1)_{L^2(\Omega_1)^n} + (\mathbf{f}_2(t), \mathbf{v}_2)_{L^2(\Omega_2)^n} \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V}, \text{ a.e. } t \in (0, T), \tag{2.22}$$

$$\left(\frac{\partial \theta_1}{\partial t}, \xi_1\right)_{L^2(\Omega_1)} + \left(\frac{\partial \theta_2}{\partial t}, \xi_2\right)_{L^2(\Omega_2)} + (k_1 \nabla \theta_1(t), \nabla \xi_1)_{L^2(\Omega_1)^n} + (k_2 \nabla \theta_2(t), \nabla \xi_2)_{L^2(\Omega_2)^n} = (\varphi_1(\boldsymbol{\sigma}_1(t), \varepsilon(\mathbf{u}_1(t)), \theta_1(t)), \xi_1)_{L^2(\Omega_1)} + (\varphi_2(\boldsymbol{\sigma}_2(t), \varepsilon(\mathbf{u}_2(t)), \theta_2(t)), \xi_2)_{L^2(\Omega_2)} + (r_1(t), \xi_1)_{L^2(\Omega_1)} + (r_2(t), \xi_2)_{L^2(\Omega_2)} \quad \forall (\xi_1, \xi_2) \in \mathcal{W}, \text{ a.e. } t \in (0, T), \tag{2.23}$$

$$\mathbf{u}_p(0) = \mathbf{u}_{p0},\ \boldsymbol{\sigma}_p(0) = \boldsymbol{\sigma}_{p0} \text{ and } \theta_p(0) = \theta_{p0} \text{ in } \Omega_p. \tag{2.24}$$

3 Main results

We establish in this section an existence and uniqueness theorem to the problem (PV) and we prove some properties of the solution concerning the regularity and stability of the solution.

3.1 Existence and Uniqueness

Theorem 1 *Under the assumptions (2.13)-(2.20), there exists a unique solution* $\{(\mathbf{u}_1, \mathbf{u}_2), (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2), (\theta_1, \theta_2)\}$ *to problem (PV). Moreover, the solution has the regularity*

$$(\mathbf{u}_1, \mathbf{u}_2) \in L^\infty(0,T;\mathcal{V}), \tag{3.1}$$

$$\left(\frac{\partial \mathbf{u}_1}{\partial t}, \frac{\partial \mathbf{u}_2}{\partial t}\right) \in L^\infty(0,T;H_1 \times H_2), \tag{3.2}$$

$$(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) \in W^{1,\infty}(0,T;\mathcal{H}_1 \times \mathcal{H}_2), \tag{3.3}$$

$$(\theta_1, \theta_2) \in L^2(0,T;\mathcal{W}) \cap L^\infty\left(0,T;L^2(\Omega_1) \times L^2(\Omega_2)\right), \tag{3.4}$$

$$\left(\frac{\partial \theta_1}{\partial t}, \frac{\partial \theta_2}{\partial t}\right) \in L^2\left(0,T;L^2(\Omega_1) \times L^2(\Omega_2)\right). \tag{3.5}$$

The proof will be carried up by two steps. Based on classical arguments of functional analysis concerning variational problems and Banach fixed point theorem.

Proof. *First step.* Take an arbitrary

$$(\eta_1, \eta_2, \lambda_1, \lambda_2) \in X \tag{3.6}$$

where

$$X = L^2(\Omega_1)_s^{n\times n} \times L^2(\Omega_2)_s^{n\times n} \times L^2(\Omega_1) \times L^2(\Omega_2),$$

and let $\mathcal{Z}_{\eta_p}$, $p = 1,2$ be the function

$$\mathcal{Z}_{\eta_p}(t) = \int_0^t \eta_p(s)\,ds + \boldsymbol{\sigma}_{p0} - \varepsilon(\mathbf{u}_{p0}). \tag{3.7}$$

Now, we consider the following auxiliary problem.

Problem (PV$_{\text{aux}}$*).* For $p = 1,2$, find the displacement field $\mathbf{u}_{\eta_p} : \Omega_p \times (0,T) \longrightarrow \mathbb{R}^n$, the stress field $\boldsymbol{\sigma}_{\eta_p} : \Omega_p \times (0,T) \longrightarrow \mathbb{S}_n$ and the temperature $\theta_{\lambda_p} : \Omega_p \times (0,T) \longrightarrow \mathbb{R}$ such that

$$\boldsymbol{\sigma}_{\eta_p}(t) = \mathcal{A}_p\left(\boldsymbol{\varepsilon}\left(\mathbf{u}_{\eta_p}(t)\right)\right) + \mathcal{Z}_{\eta_p}(t) \quad \text{a.e. } t \in (0,T), \tag{3.8}$$

$$\left(\boldsymbol{\sigma}_{\eta_1}(t), \varepsilon(\mathbf{v}_1)\right)_{L^2(\Omega_1)_s^{n\times n}} + \left(\boldsymbol{\sigma}_{\eta_2}(t), \varepsilon(\mathbf{v}_2)\right)_{L^2(\Omega_2)_s^{n\times n}} = \int_{\Gamma_1} \mathbf{g}(t)\cdot \mathbf{v}_1 d\gamma$$
$$+ \left(\mathbf{f}_1(t), \mathbf{v}_1\right)_{L^2(\Omega_1)^n} + \left(\mathbf{f}_2(t), \mathbf{v}_2\right)_{L^2(\Omega_2)^n} \ \ \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V}, \text{ a.e. } t\in(0,T), \quad (3.9)$$

$$\left(\frac{\partial\theta_{\lambda_1}}{\partial t}, \xi_1\right)_{L^2(\Omega_1)} + \left(\frac{\partial\theta_{\lambda_2}}{\partial t}, \xi_2\right)_{L^2(\Omega_2)} + \left(k_1\nabla\theta_{\lambda_1}(t), \nabla\xi_1\right)_{L^2(\Omega_1)^n}$$
$$+ \left(k_2\nabla\theta_{\lambda_2}(t), \nabla\xi_2\right)_{L^2(\Omega_2)^n} = \left(\lambda_1(t) + r_1(t), \xi_1\right)_{L^2(\Omega_1)} + \left(\lambda_2(t) + r_2(t), \xi_2\right)_{L^2(\Omega_2)}$$
$$\forall (\xi_1, \xi_2) \in \mathcal{W}, \quad \text{a.e. } t\in(0,T), \quad (3.10)$$

$$\mathbf{u}_{\eta_p}(0) = \mathbf{u}_{p0},\ \boldsymbol{\sigma}_{\eta_p}(0) = \boldsymbol{\sigma}_{p0} \text{ and } \theta_{\lambda_p}(0) = \theta_{p0} \text{ in } \Omega. \quad (3.11)$$

Lemma 1. For all $(\eta_1, \eta_2, \lambda_1, \lambda_2) \in X$, there exists a unique solution

$$\left\{\left(\mathbf{u}_{\eta_1}, \mathbf{u}_{\eta_2}\right), \left(\boldsymbol{\sigma}_{\eta_1}, \boldsymbol{\sigma}_{\eta_2}\right), \left(\theta_{\lambda_1}, \theta_{\lambda_2}\right)\right\},$$

to the auxiliary problem $(\mathrm{PV}_{\mathrm{aux}})$ and satisfying the regularity (3.1)-(3.5).

The proof of this lemma is based on classical arguments of functional analysis concerning parabolic and elliptic equations, see for more details [2], [4] and [8].

Second step. Let us consider the operator

$$\Lambda : L^\infty(0,T;X) \longrightarrow L^\infty(0,T;X),$$

defined by

$$\Lambda\left(\eta_1(t), \eta_2(t), \lambda_1(t), \lambda_2(t)\right) = \left(\mathcal{G}_1\left(\boldsymbol{\sigma}_{\eta_1}(t), \varepsilon\left(\mathbf{u}_{\eta_1}(t)\right), \theta_{\lambda_1}(t)\right),\right.$$
$$\mathcal{G}_2\left(\boldsymbol{\sigma}_{\eta_2}(t), \varepsilon\left(\mathbf{u}_{\eta_2}(t)\right), \theta_{\lambda_2}(t)\right), \varphi_1\left(\boldsymbol{\sigma}_{\eta_1}(t), \varepsilon\left(\mathbf{u}_{\eta_1}(t)\right), \theta_{\lambda_1}(t)\right),$$
$$\left.\varphi_2\left(\boldsymbol{\sigma}_{\eta_2}(t), \varepsilon\left(\mathbf{u}_{\eta_2}(t)\right), \theta_{\lambda_2}(t)\right)\right). \quad (3.12)$$

Lemma 2. The operator Λ has a fixed point $(\eta_1^*, \eta_2^*, \lambda_1^*, \lambda_2^*) \in L^\infty(0,T;X)$.

Proof. Let $t\in(0,T)$ and consider

$$(\eta_1, \eta_2, \lambda_1, \lambda_2), (\mu_1, \mu_2, \beta_1, \beta_2) \in L^\infty(0,T;X).$$

The use of (2.13) and (2.14) permits us to find for a.e. $t\in(0,T)$

$$\left\|\Lambda\left(\eta_1(t), \eta_2(t), \lambda_1(t), \lambda_2(t)\right) - \Lambda\left(\mu_1(t), \mu_2(t), \beta_1(t), \beta_2(t)\right)\right\|_X$$
$$\leq 2\left(L_{\mathcal{G}_1} + L_{\mathcal{G}_2} + L_{\varphi_1} + L_{\varphi_2}\right)\left[\left\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\mu_1}(t)\right\|_{H_1} + \left\|\mathbf{u}_{\eta_2}(t) - \mathbf{u}_{\mu_2}(t)\right\|_{H_2}\right.$$
$$+ \left\|\frac{\partial\mathbf{u}_{\eta_1}}{\partial t} - \frac{\partial\mathbf{u}_{\mu_1}}{\partial t}\right\|_{H_1} + \left\|\frac{\partial\mathbf{u}_{\eta_2}}{\partial t} - \frac{\partial\mathbf{u}_{\mu_2}}{\partial t}\right\|_{H_2}$$
$$+ \left\|\boldsymbol{\sigma}_{\eta_1}(t) - \boldsymbol{\sigma}_{\mu_1}(t)\right\|_{L^2(\Omega_1)_s^{n\times n}} + \left\|\boldsymbol{\sigma}_{\eta_2}(t) - \boldsymbol{\sigma}_{\mu_2}(t)\right\|_{L^2(\Omega_2)_s^{n\times n}}$$
$$\left.+ \left\|\theta_{\lambda_1}(t) - \theta_{\beta_1}(t)\right\|_{L^2(\Omega_1)} + \left\|\theta_{\lambda_2}(t) - \theta_{\beta_2}(t)\right\|_{L^2(\Omega_2)}\right]. \quad (3.13)$$

Taking into account equations (3.8) and (3.9), we deduce by exploiting the function $(\eta_1, \eta_2, \lambda_1, \lambda_2)$ and $(\mu_1, \mu_2, \beta_1, \beta_2)$ and choosing

$$(\mathbf{v}_1, \mathbf{v}_2) = \left(\mathbf{u}_{\eta_1} - \mathbf{u}_{\mu_1}, \mathbf{u}_{\eta_2} - \mathbf{u}_{\mu_2}\right)$$

as test function that

$$\begin{aligned}
&\left(\mathcal{A}_1\left(\varepsilon\left(\mathbf{u}_{\eta_1}(t)\right)\right) - \mathcal{A}_1\left(\varepsilon\left(\mathbf{u}_{\mu_1}(t)\right)\right), \varepsilon\left(\mathbf{u}_{\eta_1}(t)\right) - \varepsilon\left(\mathbf{u}_{\mu_1}(t)\right)\right)_{L^2(\Omega_1)_s^{n\times n}} \\
&+ \left(\mathcal{A}_2\left(\varepsilon\left(\mathbf{u}_{\eta_2}(t)\right)\right) - \mathcal{A}_2\left(\varepsilon\left(\mathbf{u}_{\mu_2}(t)\right)\right), \varepsilon\left(\mathbf{u}_{\eta_2}(t)\right) - \varepsilon\left(\mathbf{u}_{\mu_2}(t)\right)\right)_{L^2(\Omega_2)_s^{n\times n}} \\
&= -\left(\mathcal{Z}_{\eta_1}(t) - \mathcal{Z}_{\mu_1}(t), \varepsilon\left(\mathbf{u}_{\eta_1}(t)\right) - \varepsilon\left(\mathbf{u}_{\mu_1}(t)\right)\right)_{L^2(\Omega_1)_s^{n\times n}} \\
&\quad - \left(\mathcal{Z}_{\eta_2}(t) - \mathcal{Z}_{\mu_2}(t), \varepsilon\left(\mathbf{u}_{\eta_2}(t)\right) - \varepsilon\left(\mathbf{u}_{\mu_2}(t)\right)\right)_{L^2(\Omega_2)_s^{n\times n}}.
\end{aligned}$$

Which implies, keeping in mind Korn's inequality, (2.13) and (3.7) that for a.e. $t \in (0, T)$

$$\begin{aligned}
&\left\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\mu_1}(t)\right\|_{H_1} + \left\|\mathbf{u}_{\eta_2}(t) - \mathbf{u}_{\mu_2}(t)\right\|_{H_2} \leq \\
&\frac{1}{C_0^2 \min\left(\alpha_{\mathcal{A}_1}, \alpha_{\mathcal{A}_2}\right)} \int_0^t \Big(\left\|\eta_1(s) - \mu_1(s)\right\|_{L^2(\Omega_1)_s^{n\times n}} \\
&\qquad + \left\|\eta_2(s) - \mu_2(s)\right\|_{L^2(\Omega_2)_s^{n\times n}}\Big)\, ds.
\end{aligned} \tag{3.14}$$

Therefore, equation (3.8) gives for a.e. $t \in (0, T)$, using (2.20), (3.7) and (3.14)

$$\begin{aligned}
&\left\|\boldsymbol{\sigma}_{\eta_1}(t) - \boldsymbol{\sigma}_{\mu_1}(t)\right\|_{L^2(\Omega_1)_s^{n\times n}} + \left\|\boldsymbol{\sigma}_{\eta_2}(t) - \boldsymbol{\sigma}_{\mu_2}(t)\right\|_{L^2(\Omega_2)_s^{n\times n}} \leq \\
&\left(\frac{\max\left(m_{\mathcal{A}_1}, m_{\mathcal{A}_2}\right)}{C_0^2 \min\left(\alpha_{\mathcal{A}_1}, \alpha_{\mathcal{A}_2}\right)} + 1\right) \int_0^t \Big(\left\|\eta_1(s) - \mu_1(s)\right\|_{L^2(\Omega_1)_s^{n\times n}} \\
&\qquad + \left\|\eta_2(s) - \mu_2(s)\right\|_{L^2(\Omega_2)_s^{n\times n}}\Big)\, ds.
\end{aligned} \tag{3.15}$$

Furthermore, we know that the bilinear form

$$\begin{gathered}
\mathcal{W} \times \mathcal{W} \longrightarrow \mathbb{R}, \\
\left(\left(\theta_1, \theta_2\right), \left(\xi_1, \xi_2\right)\right) \longmapsto \left(k_1 \nabla\theta_1, \nabla\xi_1\right)_{L^2(\Omega_1)^n} + \left(k_2 \nabla\theta_1, \nabla\xi_2\right)_{L^2(\Omega_2)^n},
\end{gathered}$$

is not $\mathcal{W}-$elliptic, to this aim, we introduce the following function $\tilde{\xi} = e^{-t}\xi$ $\forall \xi \in H^1(\Omega_p)$, $p = 1, 2$.

By using this function in (3.10), it follows that for a.e. $t \in (0, T)$

$$\begin{aligned}
&\left(\frac{\partial\tilde{\theta}_{\lambda_1}}{\partial t}, \xi_1\right)_{L^2(\Omega_1)} + \left(\frac{\partial\tilde{\theta}_{\lambda_2}}{\partial t}, \xi_2\right)_{L^2(\Omega_2)} + \left(k_1 \nabla\tilde{\theta}_{\lambda_1}(t), \nabla\xi_1\right)_{L^2(\Omega_1)^n} \\
&+ \left(k_2 \nabla\tilde{\theta}_{\lambda_2}(t), \nabla\xi_2\right)_{L^2(\Omega_2)^n} + \left(\tilde{\theta}_{\lambda_1}(t), \xi_1\right)_{L^2(\Omega_1)} + \left(\tilde{\theta}_{\lambda_2}(t), \xi_2\right)_{L^2(\Omega_2)} = \\
&e^{-t}\left(\lambda_1(t) + r_1(t), \xi_1\right)_{L^2(\Omega_1)} + e^{-t}\left(\lambda_2(t) + r_2(t), \xi_2\right)_{L^2(\Omega_2)} \quad \forall \left(\xi_1, \xi_2\right) \in \mathcal{W},
\end{aligned}$$

and

$$\left(\frac{\partial\tilde{\theta}_{\beta_1}}{\partial t},\xi_1\right)_{L^2(\Omega_1)}+\left(\frac{\partial\tilde{\theta}_{\beta_2}}{\partial t},\xi_2\right)_{L^2(\Omega_2)}+\left(k_1\nabla\tilde{\theta}_{\beta_1}(t),\nabla\xi_1\right)_{L^2(\Omega_1)^n}$$
$$+\left(k_2\nabla\tilde{\theta}_{\beta_2}(t),\nabla\xi_2\right)_{L^2(\Omega_2)^n}+\left(\tilde{\theta}_{\beta_1}(t),\xi_1\right)_{L^2(\Omega_1)}+\left(\tilde{\theta}_{\beta_2}(t),\xi_2\right)_{L^2(\Omega_2)}=$$
$$e^{-t}\left(\beta_1(t)+r_1(t),\xi_1\right)_{L^2(\Omega_1)}+e^{-t}\left(\beta_2(t)+r_2(t),\xi_2\right)_{L^2(\Omega_2)}\ \forall(\xi_1,\xi_2)\in\mathcal{W}.$$

These yield, by subtracting the two equations, setting

$$(\xi_1,\xi_2)=\left(\tilde{\theta}_{\lambda_1}-\tilde{\theta}_{\beta_1},\tilde{\theta}_{\lambda_2}-\tilde{\theta}_{\beta_2}\right)$$

as test function and integrating over the interval time $(0,t)$

$$\frac{1}{2}\left\|\tilde{\theta}_{\lambda_1}(t)-\tilde{\theta}_{\beta_1}(t)\right\|^2_{L^2(\Omega_1)}+\frac{1}{2}\left\|\tilde{\theta}_{\lambda_2}(t)-\tilde{\theta}_{\beta_2}(t)\right\|^2_{L^2(\Omega_2)}$$
$$+\min(1,k_1^*,k_2^*)\int_0^t\left(\left\|\tilde{\theta}_{\lambda_1}(s)-\tilde{\theta}_{\beta_1}(s)\right\|^2_{H^1(\Omega_1)}+\left\|\tilde{\theta}_{\lambda_2}(s)-\tilde{\theta}_{\beta_2}(s)\right\|^2_{H^1(\Omega_2)}\right)ds$$
$$\leq\int_0^t\left(\|\lambda_1(s)-\beta_1(s)\|_{L^2(\Omega_1)}+\|\lambda_2(s)-\beta_2(s)\|_{L^2(\Omega_2)}\right)$$
$$\times\left(\left\|\tilde{\theta}_{\lambda_1}(s)-\tilde{\theta}_{\beta_1}(s)\right\|_{L^2(\Omega_1)}+\left\|\tilde{\theta}_{\lambda_2}(s)-\tilde{\theta}_{\beta_2}(s)\right\|_{L^2(\Omega_2)}\right)ds\text{ a.e. }t\in(0,T).$$

Hence, we obtain after some manipulations

$$\frac{1}{2}\left(\left\|\theta_{\lambda_1}(t)-\theta_{\beta_1}(t)\right\|_{L^2(\Omega_1)}+\left\|\theta_{\lambda_2}(t)-\theta_{\beta_2}(t)\right\|_{L^2(\Omega_2)}\right)^2$$
$$\leq\int_0^t\left(\left\|\theta_{\lambda_1}(s)-\theta_{\beta_1}(s)\right\|_{L^2(\Omega_1)}+\left\|\theta_{\lambda_2}(s)-\theta_{\beta_2}(s)\right\|_{L^2(\Omega_2)}\right)^2ds$$
$$2e^T\int_0^t\left(\|\lambda_1(s)-\beta_1(s)\|_{L^2(\Omega_1)}+\|\lambda_2(s)-\beta_2(s)\|_{L^2(\Omega_2)}\right)$$
$$\times\left(\left\|\theta_{\lambda_1}(s)-\theta_{\beta_1}(s)\right\|_{L^2(\Omega_1)}+\left\|\theta_{\lambda_2}(s)-\theta_{\beta_2}(s)\right\|_{L^2(\Omega_2)}\right)ds\text{ a.e. }t\in(0,T).$$

We deduce via Gronwall's lemma

$$\left\|\theta_{\lambda_1}(t)-\theta_{\beta_1}(t)\right\|_{L^2(\Omega_1)}+\left\|\theta_{\lambda_2}(t)-\theta_{\beta_2}(t)\right\|_{L^2(\Omega_2)}\leq$$
$$2e^{3T}\int_0^t\left(\|\lambda_1(s)-\beta_1(s)\|_{L^2(\Omega_1)}+\|\lambda_2(s)-\beta_2(s)\|_{L^2(\Omega_2)}\right)ds$$
$$\text{a.e. }t\in(0,T).\tag{3.16}$$

We conclude from (3.13)-(3.16) that there exists a constant $C > 0$ such that

$$\begin{aligned}
&\|\Lambda(\eta_1(t), \eta_2(t), \lambda_1(t), \lambda_2(t)) - \Lambda(\mu_1(t), \mu_2(t), \beta_1(t), \beta_2(t))\|_X \\
&\leq C\int_0^t \Big[\|\eta_1(s) - \mu_1(s)\|_{L^2(\Omega_1)_s^{n\times n}} + \|\eta_2(s) - \mu_2(s)\|_{L^2(\Omega_2)_s^{n\times n}} \\
&+ \|\lambda_1(s) - \beta_1(s)\|_{L^2(\Omega_1)} + \|\lambda_2(s) - \beta_2(s)\|_{L^2(\Omega_2)}\Big]\, ds \text{ a.e. } t \in (0, T),
\end{aligned}$$

This implies that

$$\begin{aligned}
&\|\Lambda(\eta_1, \eta_2, \lambda_1, \lambda_2) - \Lambda(\mu_1, \mu_2, \beta_1, \beta_2)\|_{L^\infty(0,T;X)} \\
&\leq CT\,\|(\eta_1 - \mu_1, \eta_2 - \mu_2, \lambda_1 - \beta_1, \lambda_2 - \beta_2)\|_{L^\infty(0,T;X)}. \qquad (3.17)
\end{aligned}$$

Applying Λ an other time, by recurrence on n, we obtain the following formula, see [**?**]

$$\begin{aligned}
&\|\Lambda^n(\eta_1, \eta_2, \lambda_1, \lambda_2) - \Lambda^n(\mu_1, \mu_2, \beta_1, \beta_2)\|_{L^\infty(0,T;X)} \\
&\leq \frac{C^nT^n}{n!}\|(\eta_1 - \mu_1, \eta_2 - \mu_2, \lambda_1 - \beta_1, \lambda_2 - \beta_2)\|_{L^\infty(0,T;X)}. \qquad (3.18)
\end{aligned}$$

We know that the real sequence $\left(\frac{C^nT^n}{n!}\right)_n$ converges to 0. So, for n sufficiently large $\frac{C^nT^n}{n!} < 1$. It means that a large power n of the operator Λ is a contraction on $L^\infty(0, T; X)$. Hence, Banach fixed point theorem proves that Λ^n admits a fixed point $(\eta_1^*, \eta_2^*, \lambda_1^*, \lambda_2^*) \in L^\infty(0, T; X)$.

Applying Λ, we can easily find

$$\Lambda^n(\Lambda(\eta_1^*, \eta_2^*, \lambda_1^*, \lambda_2^*)) = \Lambda(\eta_1^*, \eta_2^*, \lambda_1^*, \lambda_2^*).$$

Hence, the uniqueness of fixed point leads to

$$\Lambda(\eta_1^*, \eta_2^*, \lambda_1^*, \lambda_2^*) = (\eta_1^*, \eta_2^*, \lambda_1^*, \lambda_2^*).$$

Which permits us to conclude the proof of Lemma 2.

We can then prove the existence of solution of problem (PV). To this aim, we have

$$\begin{cases}
\mathcal{G}_1\left(\boldsymbol{\sigma}_{\eta_1^*}(t), \varepsilon\left(\mathbf{u}_{\eta_1^*}(t)\right), \theta_{\lambda_1^*}(t)\right) = \eta_1^*(t), \\
\mathcal{G}_2\left(\boldsymbol{\sigma}_{\eta_2^*}(t), \varepsilon\left(\mathbf{u}_{\eta_2^*}(t)\right), \theta_{\lambda_2^*}(t)\right) = \eta_2^*(t) \\
\varphi_1\left(\boldsymbol{\sigma}_{\eta_1^*}(t), \varepsilon\left(\mathbf{u}_{\eta_1^*}(t)\right), \theta_{\lambda_1^*}(t)\right) = \lambda_1^*(t), \\
\varphi_2\left(\boldsymbol{\sigma}_{\eta_2^*}(t), \varepsilon\left(\mathbf{u}_{\eta_2^*}(t)\right), \theta_{\lambda_2^*}(t)\right) = \lambda_2^*(t),
\end{cases}$$

and equation (3.8) can be written

$$\boldsymbol{\sigma}_{\eta_p^*}(t) = \mathcal{A}_p\left(\varepsilon\left(\mathbf{u}_{\eta_p^*}(t)\right)\right) + \mathcal{Z}_{\eta_p^*}(t), \; p = 1, 2 \text{ a.e. } t \in (0, T).$$

By derivation with respect to time variable t and using (3.7), we find for a.e. $t \in (0,T)$

$$\frac{\partial \boldsymbol{\sigma}_{\eta_p^*}}{\partial t} = \mathcal{A}_p\left(\varepsilon\left(\frac{\partial \mathbf{u}_{\eta_p^*}}{\partial t}\right)\right) + \mathcal{G}_p\left(\boldsymbol{\sigma}_{\eta_p^*}(t), \varepsilon\left(\mathbf{u}_{\eta_p^*}(t)\right), \theta_{\lambda_p^*}(t)\right),\ p = 1,2.$$

This achieves the proof. ■

3.2 Regularity of the Solution

Theorem 2 *Let the assumptions (2.13)-(2.20) hold. Then*

$$(\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{C}^0(0,T; H_1 \times H_2), \tag{3.19}$$
$$(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) \in \mathcal{C}^0(0,T; \mathcal{H}_1 \times \mathcal{H}_2), \tag{3.20}$$
$$(\theta_1, \theta_2) \in \mathcal{C}^0\left(0,T; L^2(\Omega_1) \times L^2(\Omega_2)\right), \tag{3.21}$$

In addition, if we assume that

$$\mathbf{f}_p \in \mathcal{C}^1\left(0,T; L^2(\Omega_p)^n\right),\ p = 1,2. \tag{3.22}$$

Then

$$(\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{C}^1(0,T; H_1 \times H_2), \tag{3.23}$$
$$(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) \in \mathcal{C}^1(0,T; \mathcal{H}_1 \times \mathcal{H}_2). \tag{3.24}$$

Proof. To this end, we use the theorem of intermediary derivates, see [9]. The regularity (3.1)-(3.5) gives, in particular

$$(\mathbf{u}_1, \mathbf{u}_2),\ \left(\frac{\partial \mathbf{u}_1}{\partial t}, \frac{\partial \mathbf{u}_2}{\partial t}\right) \in L^\infty(0,T; H_1 \times H_2),,$$
$$(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2), \left(\frac{\partial \boldsymbol{\sigma}_1}{\partial t}, \frac{\partial \boldsymbol{\sigma}_2}{\partial t}\right) \in L^\infty(0,T; \mathcal{H}_1 \times \mathcal{H}_2),$$
$$(\theta_1, \theta_2),\ \left(\frac{\partial \theta_1}{\partial t}, \frac{\partial \theta_2}{\partial t}\right) \in L^2\left(0,T; L^2(\Omega_1) \times L^2(\Omega_2)\right).$$

Consequently, the theorem of intermediary derivates asserts that, after a possible modification on a set of measure zero, we obtain

$$(\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{C}^0(0,T; H_1 \times H_2),$$
$$(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) \in \mathcal{C}^0(0,T; \mathcal{H}_1 \times \mathcal{H}_2),$$
$$(\theta_1, \theta_2) \in \mathcal{C}^0\left(0,T; L^2(\Omega_1) \times L^2(\Omega_2)\right).$$

Moreover, equation (2.2) implies, using hypothesis (3.22)

$$Div\left(\boldsymbol{\sigma}_p\right) \in \mathcal{C}^1\left(0, T; L^2\left(\Omega_p\right)^n\right), \ p = 1, 2.$$

This gives that

$$\left(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\right) \in \mathcal{C}^1\left(0, T; \mathcal{H}_1 \times \mathcal{H}_2\right).$$

Thus, we find using equation (2.21)

$$\mathcal{A}_p\left(\varepsilon\left(\frac{\partial \mathbf{u}_p}{\partial t}\right)\right) + \mathcal{G}_p\left(\boldsymbol{\sigma}_p, \varepsilon\left(\mathbf{u}_p\right), \theta_p\right) \in \mathcal{C}^0\left(0, T; \mathcal{H}_p\right), \ p = 1, 2. \tag{3.25}$$

On the other hand, hypothesis (2.14) implies that for $p = 1, 2$

$$\begin{aligned} &\left\|\mathcal{G}_p\left(\boldsymbol{\sigma}_p, \varepsilon\left(\mathbf{u}_p\right), \theta_p\right)\right\|_{\mathcal{C}^0\left(0,T;L^2(\Omega_p)_s^{n\times n}\right)} \le \left\|\mathcal{G}_p\left(\mathbf{0}, \mathbf{0}, 0\right)\right\|_{L^2(\Omega_p)_s^{n\times n}} \\ &+ L_{\mathcal{G}_p}\left(\left\|\boldsymbol{\sigma}_p\right\|_{\mathcal{C}^0(0,T;\mathcal{H}_p)} + \left\|\mathbf{u}_p\right\|_{\mathcal{C}^0(0,T;H_p)} + \left\|\theta_p\right\|_{\mathcal{C}^0(0,\ T;\ L^2(\Omega_p))}\right). \end{aligned} \tag{3.26}$$

By (3.25) and (3.26) we obtain

$$\left(\mathcal{A}_1\left(\varepsilon\left(\frac{\partial \mathbf{u}_1}{\partial t}\right)\right), \mathcal{A}_2\left(\varepsilon\left(\frac{\partial \mathbf{u}_2}{\partial t}\right)\right)\right) \in \mathcal{C}^0\left(0, T; L^2\left(\Omega_1\right)_s^{n\times n} \times L^2\left(\Omega_2\right)_s^{n\times n}\right).$$

Which leads to

$$\left(\frac{\partial \mathbf{u}_1}{\partial t}, \frac{\partial \mathbf{u}_2}{\partial t}\right) \in \mathcal{C}^0\left(0, T; H_1 \times H_2\right).$$

This permits us to conclude the proof. ■

3.3 Stability of the Solution

Theorem 3 *Let (2.13)-(2.16), (2.18) and (2.20) hold and let* $\left\{\mathbf{u}_p^i, \boldsymbol{\sigma}_p^i, \theta_p^i\right\}$ *be the solution of problem (PV) for the data* $\mathbf{u}_{p0}^i$, $\boldsymbol{\sigma}_{p0}^i$, θ_{p0}^i, p, $i = 1, 2$, *such that (2.17) and (2.19) hold. Then, there exists* $C > 0$ *such that*

$$\begin{aligned} &\left\|\mathbf{u}_1^2 - \mathbf{u}_1^1\right\|_{\mathcal{C}^0(0,T;H_1)} + \left\|\mathbf{u}_2^2 - \mathbf{u}_2^1\right\|_{\mathcal{C}^0(0,T;H_2)} + \left\|\boldsymbol{\sigma}_1^2 - \boldsymbol{\sigma}_1^1\right\|_{\mathcal{C}^0(0,T;\mathcal{H}_1)} \\ &\left\|\boldsymbol{\sigma}_2^2 - \boldsymbol{\sigma}_2^1\right\|_{\mathcal{C}^0(0,T;\mathcal{H}_2)} + \left\|\theta_1^2 - \theta_1^1\right\|_{\mathcal{C}^0(0,T;L^2(\Omega_1))} + \left\|\theta_2^2 - \theta_2^1\right\|_{\mathcal{C}^0(0,T;L^2(\Omega_2))} \\ &\le C\left\{\left\|\mathbf{u}_{10}^2 - \mathbf{u}_{10}^1\right\|_{H_1} + \left\|\mathbf{u}_{20}^2 - \mathbf{u}_{20}^1\right\|_{H_2} + \left\|\boldsymbol{\sigma}_{10}^2 - \boldsymbol{\sigma}_{10}^1\right\|_{\mathcal{H}_1} + \left\|\boldsymbol{\sigma}_{20}^2 - \boldsymbol{\sigma}_{20}^1\right\|_{\mathcal{H}_2}\right. \\ &\left. + \left\|\theta_{10}^2 - \theta_{10}^1\right\|_{H^1(\Omega_1)} + \left\|\theta_{20}^2 - \theta_{20}^1\right\|_{H^1(\Omega_2)}\right\} \end{aligned} \tag{3.27}$$

Proof. If $\left\{\mathbf{u}_p^i, \boldsymbol{\sigma}_p^i, \theta_p^i\right\}$ is the solution of problem (PV) for the data $\mathbf{u}_{p0}^i$, $\boldsymbol{\sigma}_{p0}^i$, θ_{p0}^i, p, $i = 1$, 2 then

$$\frac{\partial \boldsymbol{\sigma}_p^i}{\partial t} = \mathcal{A}_p\left(\varepsilon\left(\frac{\partial \mathbf{u}_p^i}{\partial t}\right)\right) + \mathcal{G}_p\left(\boldsymbol{\sigma}_p^i(t), \varepsilon\left(\mathbf{u}_p^i(t)\right), \theta_p^i(t)\right) \text{ a.e. } t \in (0,T), \tag{3.28}$$

$$\begin{gathered}\left(\boldsymbol{\sigma}_1^i(t), \varepsilon(\mathbf{v}_1)\right)_{L^2(\Omega_1)_s^{n\times n}} + \left(\boldsymbol{\sigma}_2^i(t), \varepsilon(\mathbf{v}_2)\right)_{L^2(\Omega_2)_s^{n\times n}} = \int_{\Gamma_1} \mathbf{g}(t)\cdot \mathbf{v}_1 d\gamma + \\ (\mathbf{f}_1(t), \mathbf{v}_1)_{L^2(\Omega_1)^n} + (\mathbf{f}_2(t), \mathbf{v}_2)_{L^2(\Omega_2)^n} \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V}, \\ \text{a.e. } t \in (0,T),\end{gathered} \tag{3.29}$$

$$\begin{gathered}\left(\frac{\partial \theta_1^i}{\partial t}, \xi_1\right)_{L^2(\Omega_1)} + \left(\frac{\partial \theta_2^i}{\partial t}, \xi_2\right)_{L^2(\Omega_2)} + \left(k_1 \nabla \theta_1^i(t), \nabla \xi_1\right)_{L^2(\Omega_1)^n} \\ + \left(k_2 \nabla \theta_2^i(t), \nabla \xi_2\right)_{L^2(\Omega_2)^n} = \left(\varphi_1\left(\boldsymbol{\sigma}_1^i(t), \varepsilon\left(\mathbf{u}_1^i(t)\right), \theta_1^i(t)\right), \xi_1\right)_{L^2(\Omega_1)} + \\ \left(\varphi_2\left(\boldsymbol{\sigma}_2^i(t), \varepsilon\left(\mathbf{u}_2^i(t)\right), \theta_2^i(t)\right), \xi_2\right)_{L^2(\Omega_2)} + (r_1(t), \xi_1)_{L^2(\Omega_1)} + (r_2(t), \xi_2)_{L^2(\Omega_2)} \\ \forall (\xi_1, \xi_2) \in \mathcal{W}, \quad \text{a.e. } t \in (0,T),\end{gathered} \tag{3.30}$$

$$\mathbf{u}_p^i(0) = \mathbf{u}_{p0}^i, \ \boldsymbol{\sigma}_p(0) = \boldsymbol{\sigma}_{p0}^i \text{ and } \theta_p^i(0) = \theta_{p0}^i \text{ in } \Omega_p. \tag{3.31}$$

Substituting $i = 1, 2$ in the equation (3.29), subtracting the two obtained equations, deriving the result with respect to the time variable t and choosing $(\mathbf{v}_1, \mathbf{v}_2) = \left(\frac{\partial \mathbf{u}_1^2}{\partial t} - \frac{\partial \mathbf{u}_1^1}{\partial t}, \frac{\partial \mathbf{u}_2^2}{\partial t} - \frac{\partial \mathbf{u}_2^1}{\partial t}\right)$ as test function, we obtain, using (3.28), Korn's inequality, (2.13) and (2.14)

$$\begin{gathered}\left\|\frac{\partial \mathbf{u}_1^2}{\partial t} - \frac{\partial \mathbf{u}_1^1}{\partial t}\right\|_{H_1} + \left\|\frac{\partial \mathbf{u}_2^2}{\partial t} - \frac{\partial \mathbf{u}_2^1}{\partial t}\right\|_{H_2} \leq \frac{2\max\left(L_{\mathcal{G}_1}, L_{\mathcal{G}_2}\right)}{C_0^2 \min\left(\alpha_{\mathcal{A}_1}, \alpha_{\mathcal{A}_2}\right)} \left\{\left\|\mathbf{u}_1^2(t) - \mathbf{u}_1^1(t)\right\|_{H_1}\right. \\ + \left\|\mathbf{u}_2^2(t) - \mathbf{u}_2^1(t)\right\|_{H_2} + \left\|\boldsymbol{\sigma}_1^2(t) - \boldsymbol{\sigma}_1^1(t)\right\|_{L^2(\Omega_1)_s^{n\times n}} + \left\|\boldsymbol{\sigma}_2^2(t) - \boldsymbol{\sigma}_2^1(t)\right\|_{L^2(\Omega_2)_s^{n\times n}} \\ \left. + \left\|\theta_1^2(t) - \theta_1^1(t)\right\|_{L^2(\Omega_1)} + \left\|\theta_2^2(t) - \theta_2^1(t)\right\|_{L^2(\Omega_2)}\right\} \text{ a.e. } t \in (0,T).\end{gathered} \tag{3.32}$$

Now, substituting $i = 1, 2$ in the equation (3.30), subtracting the two obtained equations and choosing $(\xi_1, \xi_2) = \left(\theta_1^2 - \theta_1^1, \theta_2^2 - \theta_2^1\right)$ as test function, it follows, using (3.15), Gronwall's lemma and some manipulations

$$\begin{gathered}\left\|\theta_1^2(t) - \theta_1^1(t)\right\|_{L^2(\Omega_1)} + \left\|\theta_2^2(t) - \theta_2^1(t)\right\|_{L^2(\Omega_2)} \leq \\ \sqrt{2} e^{2T\max\left(L_{\varphi_1}, L_{\varphi_2}\right)} \left(\left\|\theta_{10}^2 - \theta_{10}^1\right\|_{L^2(\Omega_1)} + \left\|\theta_{20}^2 - \theta_{20}^1\right\|_{L^2(\Omega_2)}\right) \\ + 2\max\left(L_{\varphi_1}, L_{\varphi_2}\right) e^{2T\max\left(L_{\varphi_1}, L_{\varphi_2}\right)} \int_0^t \left\{\left\|\mathbf{u}_1^2(s) - \mathbf{u}_1^1(s)\right\|_{H_1} + \left\|\mathbf{u}_2^2(s) - \mathbf{u}_2^1(s)\right\|_{H_2}\right. \\ \left. + \left\|\boldsymbol{\sigma}_1^2(s) - \boldsymbol{\sigma}_1^1(s)\right\|_{L^2(\Omega_1)_s^{n\times n}} + \left\|\boldsymbol{\sigma}_2^2(s) - \boldsymbol{\sigma}_2^1(s)\right\|_{L^2(\Omega_2)_s^{n\times n}}\right\} ds \\ \text{a.e. } t \in (0,T).\end{gathered} \tag{3.33}$$

Furthermore, the inequality

$$\left\|\mathbf{u}_1^2(t)-\mathbf{u}_1^1(t)\right\|_{H_1}+\left\|\mathbf{u}_2^2(t)-\mathbf{u}_2^1(t)\right\|_{H_2}\le\left\|\mathbf{u}_{10}^2-\mathbf{u}_{10}^1\right\|_{H_1}+\left\|\mathbf{u}_{20}^2-\mathbf{u}_{20}^1\right\|_{H_2}$$
$$+\int_0^t\left(\left\|\frac{\partial\mathbf{u}_1^2}{\partial t}(s)-\frac{\partial\mathbf{u}_1^1}{\partial t}(s)\right\|_{H_1}+\left\|\frac{\partial\mathbf{u}_2^2}{\partial t}(s)-\frac{\partial\mathbf{u}_2^1}{\partial t}(s)\right\|_{H_2}\right)ds$$
$$\text{a.e. } t\in(0,T), \tag{3.34}$$

combined with (3.32), gives via Gronwall's lemma

$$\left\|\mathbf{u}_1^2(t)-\mathbf{u}_1^1(t)\right\|_{H_1}+\left\|\mathbf{u}_2^2(t)-\mathbf{u}_2^1(t)\right\|_{H_2}\le$$
$$e^{2T\frac{\max\left(L_{\mathcal{G}_1},L_{\mathcal{G}_2}\right)}{C_0^2\min\left(\alpha_{\mathcal{A}_1},\alpha_{\mathcal{A}_2}\right)}}\left(\left\|\mathbf{u}_{10}^2-\mathbf{u}_{10}^1\right\|_{H_1}+\left\|\mathbf{u}_{20}^2-\mathbf{u}_{20}^1\right\|_{H_2}\right)$$
$$+\frac{2\max\left(L_{\mathcal{G}_1},L_{\mathcal{G}_2}\right)}{C_0^2\min\left(\alpha_{\mathcal{A}_1},\alpha_{\mathcal{A}_2}\right)}e^{2T\frac{\max\left(L_{\mathcal{G}_1},L_{\mathcal{G}_2}\right)}{C_0^2\min\left(\alpha_{\mathcal{A}_1},\alpha_{\mathcal{A}_2}\right)}}\int_0^t\left\{\left\|\boldsymbol{\sigma}_1^2(s)-\boldsymbol{\sigma}_1^1(s)\right\|_{L^2(\Omega_1)_s^{n\times n}}\right.$$
$$+\left\|\boldsymbol{\sigma}_2^2(s)-\boldsymbol{\sigma}_2^1(s)\right\|_{L^2(\Omega_2)_s^{n\times n}}+\left\|\theta_1^2(s)-\theta_1^1(s)\right\|_{L^2(\Omega_1)}$$
$$\left.+\left\|\theta_2^2(s)-\theta_2^1(s)\right\|_{L^2(\Omega_2)}\right\}ds\ \text{a.e. } t\in(0,T). \tag{3.35}$$

Integrating, the relation (3.28) over the interval time $(0,t)$, subtracting the two obtained equations for p, $i=1,\,2$, we find via (2.14), (2.20) and Gronwall's lemma

$$\left\|\boldsymbol{\sigma}_1^2(t)-\boldsymbol{\sigma}_1^1(t)\right\|_{L^2(\Omega_1)_s^{n\times n}}+\left\|\boldsymbol{\sigma}_2^2(t)-\boldsymbol{\sigma}_2^1(t)\right\|_{L^2(\Omega_2)_s^{n\times n}}\le$$
$$e^{T\max\left(L_{\mathcal{G}_1},L_{\mathcal{G}_2}\right)}\left(\left\|\boldsymbol{\sigma}_{10}^2-\boldsymbol{\sigma}_{10}^1\right\|_{L^2(\Omega_1)_s^{n\times n}}+\left\|\boldsymbol{\sigma}_{20}^2-\boldsymbol{\sigma}_{20}^1\right\|_{L^2(\Omega_2)_s^{n\times n}}\right)$$
$$+\max\left(m_{\mathcal{A}_1},m_{\mathcal{A}_2}\right)e^{T\max\left(L_{\mathcal{G}_1},L_{\mathcal{G}_2}\right)}\left(\left\|\mathbf{u}_{10}^2-\mathbf{u}_{10}^1\right\|_{H_1}+\left\|\mathbf{u}_{20}^2-\mathbf{u}_{20}^1\right\|_{H_2}\right)$$
$$+\max\left(m_{\mathcal{A}_1},m_{\mathcal{A}_2}\right)e^{T\max\left(L_{\mathcal{G}_1},L_{\mathcal{G}_2}\right)}\left(\left\|\mathbf{u}_1^2(t)-\mathbf{u}_1^1(t)\right\|_{H_1}+\left\|\mathbf{u}_2^2(t)-\mathbf{u}_2^1(t)\right\|_{H_2}\right)$$
$$+\max\left(L_{\mathcal{G}_1},L_{\mathcal{G}_2}\right)e^{T\max\left(L_{\mathcal{G}_1},L_{\mathcal{G}_2}\right)}\int_0^t\left\{\left\|\mathbf{u}_1^2(s)-\mathbf{u}_1^1(s)\right\|_{H_1}+\left\|\mathbf{u}_2^2(s)-\mathbf{u}_2^1(s)\right\|_{H_2}\right.$$
$$\left.+\left\|\theta_1^2(s)-\theta_1^1(s)\right\|_{L^2(\Omega_1)}+\left\|\theta_2^2(s)-\theta_2^1(s)\right\|_{L^2(\Omega_2)}\right\}ds\ \text{a.e. } t\in(0,T). \tag{3.36}$$

Hence, (3.33), (3.34), (3.35) and (3.36) imply the existence of a constant $C>0$ such that for a.e. $t\in(0,T)$

$$\begin{aligned}&\left\|\mathbf{u}_1^2(t)-\mathbf{u}_1^1(t)\right\|_{H_1}+\left\|\mathbf{u}_2^2(t)-\mathbf{u}_2^1(t)\right\|_{H_2}+\left\|\boldsymbol{\sigma}_1^2(t)-\boldsymbol{\sigma}_1^1(t)\right\|_{L^2(\Omega_1)_s^{n\times n}}\\&+\left\|\boldsymbol{\sigma}_2^2(t)-\boldsymbol{\sigma}_2^1(t)\right\|_{L^2(\Omega_2)_s^{n\times n}}+\left\|\theta_1^2(t)-\theta_1^1(t)\right\|_{L^2(\Omega_1)}+\left\|\theta_2^2(t)-\theta_2^1(t)\right\|_{L^2(\Omega_2)}\leq\\&\quad C\Big\{\left\|\mathbf{u}_{10}^2-\mathbf{u}_{10}^1\right\|_{H_1}+\left\|\mathbf{u}_{20}^2-\mathbf{u}_{20}^1\right\|_{H_2}+\left\|\boldsymbol{\sigma}_{10}^2-\boldsymbol{\sigma}_{10}^1\right\|_{L^2(\Omega_1)_s^{n\times n}}\\&\quad+\left\|\boldsymbol{\sigma}_{20}^2-\boldsymbol{\sigma}_{20}^1\right\|_{L^2(\Omega_2)_s^{n\times n}}+\left\|\theta_{10}^2-\theta_{10}^1\right\|_{L^2(\Omega_1)}+\left\|\theta_{20}^2-\theta_{20}^1\right\|_{L^2(\Omega_2)}\Big\}\\&+C\int_0^t\Big\{\left\|\mathbf{u}_1^2(s)-\mathbf{u}_1^1(s)\right\|_{H_1}+\left\|\mathbf{u}_2^2(s)-\mathbf{u}_2^1(s)\right\|_{H_2}++\left\|\boldsymbol{\sigma}_1^2(s)-\boldsymbol{\sigma}_1^1(s)\right\|_{L^2(\Omega_1)_s^{n\times n}}\\&+\left\|\boldsymbol{\sigma}_2^2(t)-\boldsymbol{\sigma}_2^1(s)\right\|_{L^2(\Omega_2)_s^{n\times n}}+\left\|\theta_1^2(s)-\theta_1^1(s)\right\|_{L^2(\Omega_1)}+\left\|\theta_2^2(t)-\theta_2^1(s)\right\|_{L^2(\Omega_2)}\Big\}\,ds.\end{aligned}$$

Then, the Gronwall lemma asserts the existence of an other positive constant, still denoted by $C>0$ such that

$$\begin{aligned}&\left\|\mathbf{u}_1^2(t)-\mathbf{u}_1^1(t)\right\|_{H_1}+\left\|\mathbf{u}_2^2(t)-\mathbf{u}_2^1(t)\right\|_{H_2}+\left\|\boldsymbol{\sigma}_1^2(t)-\boldsymbol{\sigma}_1^1(t)\right\|_{L^2(\Omega_1)_s^{n\times n}}\\&+\left\|\boldsymbol{\sigma}_2^2(t)-\boldsymbol{\sigma}_2^1(t)\right\|_{L^2(\Omega_2)_s^{n\times n}}+\left\|\theta_1^2(t)-\theta_1^1(t)\right\|_{L^2(\Omega_1)}+\left\|\theta_2^2(t)-\theta_2^1(t)\right\|_{L^2(\Omega_2)}\leq\\&\quad C\Big\{\left\|\mathbf{u}_{10}^2-\mathbf{u}_{10}^1\right\|_{H_1}+\left\|\mathbf{u}_{20}^2-\mathbf{u}_{20}^1\right\|_{H_2}+\left\|\boldsymbol{\sigma}_{10}^2-\boldsymbol{\sigma}_{10}^1\right\|_{L^2(\Omega_1)_s^{n\times n}}\\&+\left\|\boldsymbol{\sigma}_{20}^2-\boldsymbol{\sigma}_{20}^1\right\|_{L^2(\Omega_2)_s^{n\times n}}+\left\|\theta_{10}^2-\theta_{10}^1\right\|_{L^2(\Omega_1)}+\left\|\theta_{20}^2-\theta_{20}^1\right\|_{L^2(\Omega_2)}\Big\}\ \text{a.e. } t\in(0,T).\end{aligned}$$

Moreover, we know that $Div\left(\boldsymbol{\sigma}_p^1\right)=Div\left(\boldsymbol{\sigma}_p^2\right)$, $p=1,2$. Consequently

$$\begin{aligned}&\left\|\mathbf{u}_1^2-\mathbf{u}_1^1\right\|_{\mathcal{C}^0(0,T;H_1)}+\left\|\mathbf{u}_2^2-\mathbf{u}_2^1\right\|_{\mathcal{C}^0(0,T;H_2)}+\left\|\boldsymbol{\sigma}_1^2-\boldsymbol{\sigma}_1^1\right\|_{\mathcal{C}^0(0,T;\mathcal{H}_1)}\\&\left\|\boldsymbol{\sigma}_2^2-\boldsymbol{\sigma}_2^1\right\|_{\mathcal{C}^0(0,T;\mathcal{H}_2)}+\left\|\theta_1^2-\theta_1^1\right\|_{\mathcal{C}^0(0,T;L^2(\Omega_1))}+\left\|\theta_2^2-\theta_2^1\right\|_{\mathcal{C}^0(0,T;L^2(\Omega_2))}\\&\leq C\Big\{\left\|\mathbf{u}_{10}^2-\mathbf{u}_{10}^1\right\|_{H_1}+\left\|\mathbf{u}_{20}^2-\mathbf{u}_{20}^1\right\|_{H_2}+\left\|\boldsymbol{\sigma}_{10}^2-\boldsymbol{\sigma}_{10}^1\right\|_{\mathcal{H}_1}+\left\|\boldsymbol{\sigma}_{20}^2-\boldsymbol{\sigma}_{20}^1\right\|_{\mathcal{H}_2}\\&\qquad+\left\|\theta_{10}^2-\theta_{10}^1\right\|_{H^1(\Omega_1)}+\left\|\theta_{20}^2-\theta_{20}^1\right\|_{H^1(\Omega_2)}\Big\}.\end{aligned}$$

Which achieves the proof. ■

4 Open Problem

The case when the dissipative function φ_p in the energy equations is not necessary Lipschitzian (for example, the viscose dissipation, which can be written as the product of the stress tensor and the plastic part of the rate of deformation tensor) remains unsolved and need several mathematical techniques, like the L^1 data theory.

Moreover, it is of interest to investigate setting with taking into account the phenomena of contact with or without friction on the transmission interface. Mathematically, these are likely to turn out to be vey hard problems. There is the possibility of thermal instability.

We notice that the processes of dynamic evolution for these rate-type constitutive laws have been never treated. New mathematical tools need to be developed for this task. Since variational methods are incapable to solve these problems, we must use numerical techniques to approximate and simulate such models.

We also notice that the transmission between two different models (like transmission between elastic and plastic or viscoelastic and viscoplastic bodies) is an open problem.

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