Int. J. Open Problems Compt. Math., Vol. 7, No. 1, March 2014
ISSN 1998-6262; Copyright ©ICSRS Publication, 2014
www.i-csrs.org

On univalence criteria for analytic functions defined by multiplier transformation and Ruscheweyh derivative

Alina Alb Lupaş

Department of Mathematics and Computer Science University of Oradea,
str. Universitatii nr. 1, 410087 Oradea, Romania
e-mail:dalb@uoradea.ro

Abstract

In this paper we obtain sufficient conditions for univalence of analytic functions defined by the linear operator $RI_{n,\lambda,l}^{\alpha}:\mathcal{A}\to\mathcal{A}$, $RI_{n,\lambda,l}^{\alpha}f(z)=(1-\alpha)R^{n}f(z)+\alpha I\left(n,\lambda,l\right)f(z)$, $z\in U$***, where*** $R^{n}f(z)$ ***is the Ruscheweyh derivative,*** $I\left(n,\lambda,l\right)$ ***the multiplier transformation and*** $\mathcal{A}_{n}=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+\ldots,\ z\in U\}$ ***is the class of normalized analytic functions with*** $\mathcal{A}_{1}=\mathcal{A}$.

Keywords: *differential operator, analytic functions, univalent functions.*
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Denote by U the unit disc of the complex plane, $U=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_{n}=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+\ldots,\ z\in U\}$ with $\mathcal{A}_{1}=\mathcal{A}$ and $\mathcal{S}$ the subclass of functions that are univalent in U.

Definition 1.1. *(Ruscheweyh [23]) For $f\in\mathcal{A}$, $n\in\mathbb{N}$, the operator R^{n} is defined by $R^{n}:\mathcal{A}\to\mathcal{A}$,*

$$\begin{aligned} R^{0}f\left(z\right) &= f\left(z\right) \\ R^{1}f\left(z\right) &= zf'\left(z\right),\ \ldots \\ \left(n+1\right)R^{n+1}f\left(z\right) &= z\left(R^{n}f\left(z\right)\right)'+nR^{n}f\left(z\right),\quad z\in U. \end{aligned}$$

Remark 1.2. *If* $f \in \mathcal{A}$*,* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$*, then* $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$*,* $z \in U$*.*

Definition 1.3. *For* $f \in \mathcal{A}$*,* $n \in \mathbb{N}$*,* $\lambda, l \geq 0$*, the operator* $I(n, \lambda, l) f(z)$ *is defined by the following infinite series*

$$I(n, \lambda, l) f(z) = z + \sum_{j=2}^{\infty} \left(\frac{\lambda(j-1) + l + 1}{l+1} \right)^n a_j z^j.$$

Remark 1.4. *It follows from the above definition that*

$$I(0, \lambda, l) f(z) = f(z),$$

$$(l+1) I(n+1, \lambda, l) f(z) = (l+1-\lambda) I(n, \lambda, l) f(z) + \lambda z (I(n, \lambda, l) f(z))',$$

$z \in U$.

Remark 1.5. *For* $l = 0$*,* $\lambda \geq 0$*, the operator* $D_\lambda^n = I(n, \lambda, 0)$ *was introduced and studied by Al-Oboudi [15], which is reduced to the Sălăgean differential operator [24] for* $\lambda = 1$*.*

Definition 1.6. *[7] Let* $\alpha, \lambda, l \geq 0$*,* $n \in \mathbb{N}$*. Denote by* $RI_{n,\lambda,l}^{\alpha}$ *the operator given by* $RI_{n,\lambda,l}^{\alpha} : \mathcal{A} \to \mathcal{A}$*,*

$$RI_{n,\lambda,l}^{\alpha} f(z) = (1-\alpha) R^n f(z) + \alpha I(n, \lambda, l) f(z), \ z \in U.$$

Remark 1.7. *If* $f \in \mathcal{A}$*,* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$*, then*
$RI_{n,\lambda,l}^{\alpha} f(z) = z + \sum_{j=2}^{\infty} \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j$*,* $z \in U$*.*
This operator was studied also in [13], [14].

Remark 1.8. *For* $\alpha = 0$*,* $RI_{m,\lambda,l}^{0} f(z) = R^m f(z)$*, where* $z \in U$ *and for* $\alpha = 1$*,* $RI_{m,\lambda,l}^{1} f(z) = I(m, \lambda, l) f(z)$*, where* $z \in U$*, which was studied in [3], [4], [10], [9]. For* $l = 0$*, we obtain* $RI_{m,\lambda,0}^{\alpha} f(z) = RD_{\lambda,\alpha}^{m} f(z)$ *which was studied in [5], [6], [11], [12], [16], [17] and for* $l = 0$ *and* $\lambda = 1$*, we obtain* $RI_{m,1,0}^{\alpha} f(z) = L_\alpha^m f(z)$ *which was studied in [1], [2], [8].*

Our considerations are based on the following results.

Lemma 1.9. *[18] Let* $f \in \mathcal{A}$*. If for all* $z \in U$

$$\left(1 - |z|^2\right) \left| \frac{z f''(z)}{f'(z)} \right| \leq 1,$$

then the function f *is univalent in* U*.*

Lemma 1.10. *[21] Let $f \in \mathcal{A}$. If for all $z \in U$*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq 1,$$

then the function f is univalent in U.

Lemma 1.11. *[25] Let μ be a real number, $\mu > \frac{1}{2}$ and $f \in \mathcal{A}$. If for all $z \in U$*

$$\left|\left(1 - |z|^{2\mu}\right)\frac{zf''(z)}{f'(z)} + 1 - \mu\right| \leq \mu,$$

then the function f is univalent in U.

Lemma 1.12. *[20]If $f(z) \in \mathcal{S}$ and*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

then

$$\sum_{n=1}^{\infty}(n-1)\left|b_n\right|^2 \leq 1.$$

Lemma 1.13. *[22] Let $\nu \in \mathbb{C}$, $Re(\nu) \geq 0$ and $f \in \mathcal{A}$. If for all $z \in U$*

$$\frac{\left(1 - |z|^{2Re(\nu)}\right)}{Re(\nu)}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

then the function

$$F_\nu(z) = \left(\nu \int_0^z u^{\nu-1} f'(u)\, du\right)^{\frac{1}{\nu}}$$

is univalent in U.

2 The main result

Following the paper of M. Darus and R. Ibrahim [19], we establish the sufficient conditions to obtain a univalence for analytic function involving the differential operator $RI^{\alpha}_{n,\lambda,l}$.

Theorem 2.1. *Let $f \in \mathcal{A}$. If for all $z \in U$,*

$$\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\alpha)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}\left[j(2j-1)\right]\left|a_j\right| \leq 1. \tag{1}$$

Then $RI^{\alpha}_{n,\lambda,l} f(z)$ is univalent in U.

Proof Let $f \in \mathcal{A}$. Assume that (1) is hold. Then for all $z \in U$ we have

$$\left(1-|z|^2\right)\frac{z\left(RI^{\alpha}_{n,\lambda,l}f(z)\right)''}{\left(RI^{\alpha}_{n,\lambda,l}f(z)\right)'} \le \left(1+|z|^2\right)\left|\frac{z\left(RI^{\alpha}_{n,\lambda,l}f(z)\right)''}{\left(RI^{\alpha}_{n,\lambda,l}f(z)\right)'}\right| =$$

$$\left(1+|z|^2\right)\left|\frac{z\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}j(j-1)a_jz^{j-2}}{\left(1+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}ja_jz^{j-1}\right)}\right| \le$$

$$\frac{2\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}j(j-1)|a_j|}{1-\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}j|a_j|} \le 1.$$

Thus, in view of Lemma 1.9, $RI^{\alpha}_{n,\lambda,l}f(z)$ is univalent in U.

Theorem 2.2. *Let $f \in \mathcal{A}$. If for all $z \in U$,*

$$\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j| \le \frac{1}{\sqrt{7}}. \qquad (2)$$

Then $RI^{\alpha}_{n,\lambda,l}f(z)$ is univalent in U.

Proof Let $f \in \mathcal{A}$. Assume that (2) is hold. It is sufficient to show that

$$\left|\frac{z^2\left(RI^{\alpha}_{n,\lambda,l}f(z)\right)'}{\left(RI^{\alpha}_{n,\lambda,l}f(z)\right)^2}-1\right| \le 1,$$

which is equivalent to show that

$$\left|\frac{z^2\left(RI^{\alpha}_{n,\lambda,l}f(z)\right)'}{2\left(RI^{\alpha}_{n,\lambda,l}f(z)\right)^2}\right| \le 1.$$

We have $\left|\frac{z^2\left(RI^{\alpha}_{n,\lambda,l}f(z)\right)'}{2\left(RI^{\alpha}_{n,\lambda,l}f(z)\right)^2}\right| = \left|\frac{z^2\left(1+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}ja_jz^{j-1}\right)}{2\left(z+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j\right)^2}\right| =$

$$\left|\frac{1+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}ja_jz^{j-1}}{2\left(1+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^{j-1}+\left(\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^{j-1}\right)^2\right)}\right|$$

$$\le \frac{1+\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}j|a_j|}{2\left(1-2\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|-\left(\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|^2\right)\right)}$$

which is less than 1 if the assertion (2) is hold. Thus in view of Lemma 1.10, $RI^{\alpha}_{n,\lambda,l}f(z)$ is univalent in U.

Theorem 2.3. *Let $f \in \mathcal{A}$. If for all $z \in U$*

$$\sum_{j=2}^{\infty} j\left[2(j-1)+(2\mu-1)\right]\cdot$$

$$\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\alpha)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}|a_j| \leq 2\mu-1, \quad \mu>\frac{1}{2}. \quad (3)$$

Then $RI_{n,\lambda,l}^{\alpha}f(z)$ is univalent in U.

Proof Let $f \in \mathcal{A}$. Then for all $z \in U$ we have

$$\left|\left(1-|z|^{2\mu}\right)\frac{z\left(RI_{n,\lambda,l}^{\alpha}f(z)\right)''}{\left(RI_{n,\lambda,l}^{\alpha}f(z)\right)'}+1-\mu\right| \leq \left(1+|z|^{2\mu}\right)\left|\frac{z\left(RI_{n,\lambda,l}^{\alpha}f(z)\right)''}{\left(RI_{n,\lambda,l}^{\alpha}f(z)\right)'}\right|+|1-\mu|$$

$$\leq \frac{2\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}j(j-1)|a_j|}{1-\sum_{j=2}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}j|a_j|}+|1-\mu|$$

the last inequality is less that μ if the assertion (3) is hold. thus, in view of Lemma 1.11, $RI_{n,\lambda,l}^{\alpha}f(z)$ is univalent in U.

As applications of Theorems 2.1, 2.2 and 2.3 we have the following result

Theorem 2.4. *Let $f \in \mathcal{A}$. If for all $z \in U$ one of the inequalities (1-3) holds, then*

$$\sum_{j=1}^{\infty}(j-1)|b_j|^2 \leq 1,$$

where

$$\frac{z}{RI_{n,\lambda,l}^{\alpha}f(z)}=1+\sum_{j=1}^{\infty}b_j z^j.$$

Proof Let $f \in \mathcal{A}$. Then, in view of Theorems 2.1, 2.2 or 2.3, $RI_{n,\lambda,l}^{\alpha}f(z)$ is univalent in U. Hence, by Lemma 1.12 we obtain the result.

Theorem 2.5. *Let $f \in \mathcal{A}$. If for all $z \in U$,*

$$\sum_{j=2}^{\infty} j\left[2(j-1)+Re(\nu)\right]\cdot$$

$$\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\alpha)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}|a_j| \leq Re(\nu), \quad Re(\nu)>0. \quad (4)$$

Then

$$G_{\nu}(z) = \left(\nu \int_0^z u^{\nu-1} \left(RI^{\alpha}_{n,\lambda,l} f(u)\right)' du\right)^{\frac{1}{\nu}}$$

is univalent in U.

Proof Let $f \in \mathcal{A}$. Then for all $z \in U$, we have

$$\frac{\left(1-|z|^{2Re(\nu)}\right)}{Re(\nu)} \left| \frac{z\left(RI^{\alpha}_{n,\lambda,l} f(z)\right)''}{\left(RI^{\alpha}_{n,\lambda,l} f(z)\right)'} \right| \le \frac{\left(1+|z|^{2Re(\nu)}\right)}{Re(\nu)} \left| \frac{z\left(RI^{\alpha}_{n,\lambda,l} f(z)\right)''}{\left(RI^{\alpha}_{n,\lambda,l} f(z)\right)'} \right| \le$$
$$\frac{2}{Re(\nu)} \frac{\sum_{j=2}^{\infty} \left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\} j(j-1)|a_j|}{1-\sum_{j=2}^{\infty} \left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\} j|a_j|}.$$

The last inequality is less than 1 if the assertion (4) is hold. Thus, in view of Lemma 1.13, $G_{\nu}(z)$ is univalent.

3 Open Problem

Find other sufficient conditions for univalence of analytic functions defined by this differential operator.

References

[1] A. Alb Lupaş, *On special differential subordinations using Sălăgean and Ruscheweyh operators*, Mathematical Inequalities and Applications, Volume 12, Issue 4, 2009, 781-790.

[2] A. Alb Lupaş, *On a certain subclass of analytic functions defined by Salagean and Ruscheweyh operators*, Journal of Mathematics and Applications, No. 31, 2009, 67-76.

[3] A. Alb Lupaş, *A special comprehensive class of analytic functions defined by multiplier transformation*, Journal of Computational Analysis and Applications, Vol. 12, No. 2, 2010, 387-395.

[4] A. Alb Lupaş, *A new comprehensive class of analytic functions defined by multiplier transformation*, Mathematical and Computer Modelling 54 (2011) 2355–2362.

[5] A. Alb Lupaş, *On special differential subordinations using a generalized Sălăgean operator and Ruscheweyh derivative*, Journal of Computational Analysis and Applications, Vol. 13, No.1, 2011, 98-107.

[6] A. Alb Lupaş, *On a certain subclass of analytic functions defined by a generalized Sălăgean operator and Ruscheweyh derivative*, Carpathian Journal of Mathematics, 28 (2012), No. 2, 183-190.

[7] A. Alb Lupaş, *On special differential subordinations using multiplier transformation and Ruscheweyh derivative*, Romai Journal, Vol. 6, Nr. 2, 2010, p. 1-14.

[8] A. Alb Lupaş, D. Breaz, *On special differential superordinations using Sălăgean and Ruscheweyh operators*, Geometric Function Theory and Applications' 2010 (Proc. of International Symposium, Sofia, 27-31 August 2010), 98-103.

[9] A. Alb Lupaş, *Certain special differential superordinations using multiplier transformation*, International Journal of Open Problems in Complex Analysis, Vol. 3, No. 2, July, 2011, 50-60.

[10] A. Alb Lupaş, *On special differential superordinations using multiplier transformation*, Journal of Computational Analysis and Applications, Vol. 13, No.1, 2011, 121-126.

[11] A. Alb Lupaş, *On special differential superordinations using a generalized Sălăgean operator and Ruscheweyh derivative*, Computers and Mathematics with Applications 61, 2011, 1048-1058, doi:10.1016/j.camwa.2010.12.055.

[12] A. Alb Lupaş, *Certain special differential superordinations using a generalized Sălăgean operator and Ruscheweyh derivative*, Analele Universitatii Oradea, Fasc. Matematica, Tom XVIII, 2011, 167-178.

[13] A. Alb Lupaş, *On special differential superordinations using multiplier transformation and Ruscheweyh derivative*, International Journal of Research and Reviews in Applied Sciences 9 (2), November 2011, 211-222.

[14] A. Alb Lupaş, *Certain special differential superordinations using multiplier transformation and Ruscheweyh derivative*, Journal of Computational Analysis and Applications, Vol. 13, No.1, 2011, 108-115.

[15] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Ind. J. Math. Math. Sci., 27 (2004), 1429-1436.

[16] L. Andrei, *Differential subordinations using Ruscheweyh derivative and generalized Sălăgean operator*, Advances in Difference Equation, 2013, 2013:252, DOI: 10.1186/1687-1847-2013-252.

[17] L. Andrei, V. Ionescu, *Some differential superordinations using Ruscheweyh derivative and generalized Sălăgean operator*, Journal of Computational Analysis and Applications, Vol. 17, No. 3, 2014, 437-444

[18] J. Becker, *Löwnersche Differential gleichung und quasi-konform fortsetzbare schlichte funktionen*, J. Reine Angew. Math., 255, (1972), 23-43.

[19] M. Darus, R. Ibrahim, *On univalence criteria for analytic functions defined by a generalized differential operator*, Acta Universitatis Apulensis, No. 23, 2010, 195-200.

[20] A. W. Goodman, *Univalent Functions*, Vol. I and II, Mariner, Tampa, Florida, 1983.

[21] S. Ozaki, M. Nunokawa, *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc., 33 (2), (1972), 392-394.

[22] N.N. Pascu, *On the univalence criterion of Becker*, Mathematica, Cluj-Napoca, Tome 29 (52), Nr. 2 (1987), 175-176.

[23] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amet. Math. Soc., 49(1975), 109-115.

[24] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013(1983), 362-372.

[25] H. Tudor, *A sufficient condition for univalence*, General Math., Vol. 17, No. 1 (2009), 89-94.