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Dual Series Method for Solving Heat Equation with Mixed Boundary Conditions

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Abstract

Paper is devoted to determine the solution of a non-stationary heat equation in an axial symmetry cylindrical coordinates subject to a nonhomogeneous mixed discontinuous boundary conditions of the first and of the second kind inside the disk of an finite surface cylinder. The solution of the given mixed boundary value problem is obtained with the aid of a classical methods and based on the application of a new type of a dual series equations (DSE) with a Bessel function of the first kind of order zero as a kernel. The solution of obtained DSE which is discussed in this paper is introduced to a Fredholm integral equation of the first kind.

Keywords: *Dual series equations, mixed boundary conditions, Fredholm integral equation of the first kind.*

1 Introduction

The use of the DS method for solving time independent mathematical physics equations for many physical and technical applications with mixed boundary conditions can be found for example monographs [12,13]and other references. DSE arises in applications of diffraction theory, stationary heat theory, electrostatic theory, elasticity theory and other areas of application with different coordinate systems. In this paper we introduce a new type of the DSE related to the time dependent homogeneous heat equation in cylindrical coordinates subject to nonhomogeneous mixed boundary conditions of the first and of the second kind located on the level surface of a bounded cylinder with constant initial condition. The construction of DSE for a given problem is based on the solution of homogeneous mixed boundary value problems and dual integral equations dealing with heat and Helmholtz equations which is discussed with details in monographs [4-7,10]. In this paper we developed the solution of the given mixed problem with the use of a Laplace transform (L-transform), separation of variables, Hankel integral transform and some discontinuous integrals. The solution of the DSE which is studied in this paper is reduced to some type of a Fredholm integral equation of the first kind with kernel, free term and unknown function depend on L-transform parameter, such integral equations should be solved by iterative methods. In particular, if the L-transform parameter tends to zero, the solution of the above mixed boundary value problem is reduced to a Fredholm integral equations of the second kind.

2 Formulation and Solution of the Problem

It is required to find a temperature distribution function

$$\theta(r,z,\tau) = T(r,z,\tau) - T_0.$$

Where $0 < r < r_0 < R,\ 0 < z < \infty,\ \tau > 0$ is the initial temperature (constant) r, z cylindrical coordinates variables, satisfies the initial mixed boundary value problem

$$\theta_{rr}(r,z,\tau) + \theta_r(r,z,\tau)/r + \theta_{zz}(r,z,\tau) = \theta_\tau(r,z,\tau)\ /a\quad , \qquad (2.1)$$

a is a heat diffusivity coefficient (constant) . On the surface of the cylinder inside the disk $z = 0,\ 0 < r < r_0 < R$, given discontinuous mixed boundary conditions of the first and of the second kind (r_0 is the line of discontinuity)

$$\theta(r,0,\tau) = f(r,\tau), \quad 0 \le r < r_0, \tag{2.2}$$

$$\theta_z(r,0,\tau) = 0, \quad r_0 < r < R \tag{2.3}$$

where $f(r,\tau)$, known continuous and integrable function with respect to two variables r,τ accept L-transform with respect to τ and Hankel integral transform with respect to r .

On $r = R$, $r = 0$, $z \to \infty$, the boundary conditions are

$$\theta(R,z,\tau) = 0, \quad 0 \le z < \infty, \tag{2.4}$$

$$\theta(0,z,\tau) = 0, \quad 0 \le z < \infty, \tag{2.5}$$

$$\theta(r,\infty,\tau) = 0, \quad 0 \le r < R \ . \tag{2.6}$$

The physical significance of the problem formulated such that, on the level surface of semi-infinite cylinder inside the disk $z = 0,\ 0 < r < r_0$, a mixed boundary condition of the first kind prescribed temperature is considered, whereas outside the disk $z = 0,\ r_0 < r < R$, a normal derivative function is zero according Fourier low (heat insulated) is given. On the lateral surface $r = R$, $z > 0$ a first kind homogeneous boundary conditions is given (zero temperature).

Now use L-transform to (2.1) to (2.6), where

$$\overline{\theta}(r,z,s) = l\left[\theta(r,z,\tau)\right] = \int_0^\infty \theta(r,z,\tau)\exp(-s\tau)d\tau$$

Then separate variables in (2.1) and use the boundary conditions(2.4)-(2.6), we obtain a general of the above problem solution in L-transform image

$$\overline{\theta}(r,z,s) = \overline{T}(r,z,s) - T_0/s$$
$$= \sum_{n=1}^{\infty} \overline{C}_n(\lambda_n,s)\exp(-|z/r_0|\ \sqrt{\lambda_n^2 + sr_0^2/a}\ )J_0(\lambda_n\rho) \tag{2.7}$$

$\overline{C}_n(\lambda_n, s)$ unknown coefficients , λ_n is the root of Bessel function of the first kind order zero $J_0(\lambda_n\alpha)=0$,moreover, $\rho = r/r_0$, $\alpha = R/r_0$. s is the parameter of L-transform.

Use a mixed conditions (2.2) and (2.3) for (2.7), we obtain a DSE to determine the unknown coefficients $\overline{C}_n(\lambda_n, s)$

$$\sum_{n=1}^{\infty}\overline{C}_n(\lambda_n, s)J_0(\lambda_n\rho) = \overline{f}(\rho, s), \quad 0 \le \rho < 1, \tag{2.8}$$

$$\sum_{n=1}^{\infty}\overline{C}_n(\lambda_n, s)\sqrt{\lambda_n^2/r_0^2 + s/a}\; J_0(\lambda_n\rho) = 0 \quad 1 < \rho < \alpha. \tag{2.9}$$

As $s \to 0$ the DSE (2.8), (2.9) were introduced to the stationary solution of the DSE involving Laplace equation with mixed conditions [13].

Now to solve (2.8) and (2.9), let us to introduce the equality

$$\sum_{n=1}^{\infty}\overline{A}_n(\lambda_n, s)\; J_0(\lambda_n\rho) = \overline{h}(\rho, s), \quad 0 \le \rho < 1, \tag{2.10}$$

where $\overline{A}_n(\lambda_n, s) = \overline{C}_n(\lambda_n, s)/\sqrt{\lambda_n^2/r_0^2 + s/a}$, $\overline{h}(\rho, s)$ is an unknown function defined over the interval $\rho \in [0,1)$ $\overline{h}(\rho, s) = L[h(\rho, \tau)]$.According to Fourier-Bessel inversion formula for a series (2.10), we have

$$\overline{A}_n(\lambda_n, s) = \frac{2}{\alpha^2 J_1^2(\lambda_n\alpha)}\int_0^1 \overline{h}(u, s)J_0(\lambda_n u)u\,du\ . \tag{2.11}$$

In (2.11) replace the unknown function $\overline{h}(\rho, s)$ by another unknown function $\overline{\phi}(t, s)$ with the help of the relation [13]

$$\overline{h}(\rho, s) = -\frac{1}{\rho}\frac{d}{d\rho}\int_\rho^1 \frac{\overline{\phi}(t, s)}{\sqrt{t^2-\rho^2}}dt\quad . \tag{2.12}$$

Substitute (2.12) in (2.11), change the order of integration, then use the equalities [11,13]

$$\int_0^t \frac{J_1(\lambda_n u)}{\sqrt{t^2-u^2}}\,du = \sqrt{\frac{\pi\lambda_n}{2t}}J_{-1/2}(\lambda_n t)$$

$$\frac{d}{du} J_0(\lambda_n u) = -\lambda_n J_{-1}(\lambda_n u)$$

Equality (2.11) is written as

$$\begin{aligned}\overline{A}_n(\lambda_n, s) &= \frac{-2}{\alpha^2 J_1^2(\lambda_n \alpha)} \int_0^1 J_0(\lambda_n u)\left[\frac{d}{du}\int_u^1 \frac{\overline{\varphi}(t,s)}{\sqrt{t^2-u^2}}\,dt\right]du \\ &= \frac{\sqrt{2\pi\lambda_n}}{\alpha^2 J_1^2(\lambda_n\alpha)} \int_0^1 \frac{\overline{\varphi}(t,s)}{\sqrt{t}} J_{-1/2}(\lambda_n t)\,dt\end{aligned} \tag{2.13}$$

Next, substitute expression (2.13) into (2.8), we obtain an integral equation of the first kind for determination $\overline{\phi}(t,s)$

$$\frac{\sqrt{2\pi}}{\alpha^2}\int_0^1 \frac{\overline{\varphi}(t,s)}{\sqrt{t}}\left[\sum_{n=1}^{\infty}\frac{\sqrt{\lambda_n}\, J_0(\lambda_n\rho) J_{-1/2}(\lambda_n t)}{J_1^2(\lambda_n\alpha)\sqrt{\lambda_n^2/r_0^2+s/a}}\right]dt = \overline{f}(\rho,s), \quad 0\le\rho<1. \tag{2.14}$$

Rewrite a Fredholm integral equation of the first kind (2.14) in the standard form

$$\int_0^1 \overline{K}(t,\rho,s)\overline{\psi}(t,s)\,dt = \overline{f}(\rho,s) \quad , \quad 0\le\rho<1 \tag{2.15}$$

$$\overline{K}(t,\rho,s) = \frac{2}{\alpha^2}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\rho)\cos(\lambda_n t)}{J_1^2(\lambda_n\alpha)\sqrt{\lambda_n^2/r_0^2+s/a}},$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos(x), \quad t\overline{\psi}(t,s) = \overline{\phi}(t,s).$$

The solution of a Fredholm integral equation of the first kind (2.15) should be treated by iterative techniques which discussed in [7, 8, and 14]. The kernel in (2.15) converges for any variables t,ρ and for value of the parameter λ_n. Solving a first order integral equation (2.15) with kernel $\overline{K}(t,\rho,s)$, by using the known iteration[14]

$$\overline{\psi}(t,s)\in L_2(0,1),\ \overline{f}(\rho,s)\in L_2(0,1)$$

$$\overline{\psi}_0(t,s) = 0$$

$$\overline{\psi_1}(t,s)=\beta\,\overline{f}(\rho,s)$$

$$\overline{\psi}_m(t,s)=\overline{\psi}_{m-1}(t,s)+\beta[\overline{f}(\rho,s)-U\overline{\psi}_{m-1}(t,s)]\,,\ m=1,2,3,... \qquad (2.16)$$

Where β constant, moreover,

$$U\overline{\psi}_m(t,s)=\int_0^1\overline{K}(t,\rho,s)\overline{\psi}_m(t,s)d\rho \qquad , \qquad \left\|U^2\right\|\le\int_0^1\int_0^1 K^2(t,\rho)\,dtd\rho$$

$\overline{\psi}_m(t,s)$ converges for any values of m,n, and satisfies the property [14]

$$\left\|\overline{\psi}(t,s)-\ \overline{\psi}_m(t,s)\right\|=\frac{\beta}{1-\left|\beta U-1\right|}<\infty,\ 0<\beta<2/\left\|U\right\|.$$

The inverse L- transform for the left hand side of (2.14) is [1]

$$L^{-1}\left[\frac{\overline{\psi}(t,s)}{\sqrt{s+a\lambda_n^2/r_0^2}}\right]=\frac{1}{\sqrt{\pi}}\int_0^\tau\frac{\psi(t,\xi)}{\sqrt{\tau-\xi}}\exp\left[-\lambda_n^2\,a(\tau-\xi)/r_0^2\right]d\xi.$$

Thus equation (2.14) becomes

$$\frac{\sqrt{2}}{\pi\alpha^2}\int_0^1\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\rho)J_{-1/2}(\lambda_n t)}{J_1^2(\lambda_n\alpha)}A_n(\tau,\xi)=f(r,\tau)\,,\quad 0\le\rho<1. \qquad (2.17)$$

Inside integral equation (2.17), we introduced the value

$$\int_0^r\frac{\psi(t,\xi)}{\sqrt{\tau-\xi}}\exp\left[-\lambda_n^2 a\,(\tau-\xi)/r_0^2\right]d\xi=A_n(t,\tau)\,. \qquad (2.18)$$

Treat (2.18) as an Abel integral equation, the function $\phi(t,\tau)$ by using inversion formula is

$$\psi(t,\tau)=\frac{1}{\pi\lambda_n}\ \frac{d}{d\tau}\int_0^r\frac{F_n(t,\xi)}{\sqrt{\tau-\xi}}\exp\left[-\lambda_n^2 a(\tau-\xi)/r_0^2\right]d\xi\ . \qquad (2.19)$$

Now if $s\to 0$ in integral (2.14), we discover that

$$\int_0^1\frac{\psi(t)}{\sqrt{t}}\,S(\rho,t)dt=f(\rho,),\ \ 0\le\rho<1, \qquad (2.20)$$

method of solution given in [13] where

$$\begin{aligned}S(\rho,t)&=\frac{2}{\alpha^2}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\rho)J_{-1/2}(\lambda_n t)}{\sqrt{\lambda_n}J_1^2(\lambda_n\alpha)}\\&=\int_0^\infty J_0(\rho x)J_{-1/2}(tx)\sqrt{x}\,dx-\frac{2}{\pi}L(\rho,t),\end{aligned} \qquad (2.21)$$

$$L\ (\rho,t)=\int_0^\infty \frac{K_0\,(\alpha\,y)}{I_0(\alpha\,y)}\,I_{-1/2}(\rho\,y)\,I_{-1/2}\,(ty)\sqrt{y}\,dy$$

$I_0\,(x)$, $K_0\,(x)$ in (2.22) are a modified Bessel functions of the second kind of order zero

Now use a well known integral [11]

$$\int_0^\infty J_0\,(\rho\,x)J_{-1/2}\,(t\,x)\sqrt{x}\,dx=\sqrt{\frac{2}{\pi\,t}}\,\frac{H\,(\rho-t)}{\sqrt{\rho^2-t^2}} \quad , \qquad (2.\ 22)$$

$$H\ (\rho-t)=\begin{cases}0, & \rho<t\\ 1, & \rho>t.\end{cases}$$

is a Heaviside unit step function.

First order integral equation (2.20) with the use of (2.21), (2.22) is reduced to an integral equation of the second kind as

$$\int_0^\rho \frac{\psi\,(t)/t}{\sqrt{t^2-\rho^2}}\,dt=\sqrt{\pi}\ f(\rho)+\sqrt{\frac{2}{\pi}}\int_0^1 \frac{\psi\ (t)}{\sqrt{t}}\ L\ (\rho,t)\,dt\,. \qquad (2.23)$$

Treat (2.23) as an Abels' integral equation, we obtain a Fredholm integral equation of the second kind for determination an unknown function $\psi(t)$

$$\begin{aligned}\psi\,(t)&=\frac{2t}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{u\,f\ (u)du}{\sqrt{t^2-u^2}}\\ &+(2/\pi)^{3/2}t\,\frac{d}{dt}\int_0^t \frac{u}{\sqrt{t^2-u^2}}\left(\int_0^1 \frac{\psi\,(y)}{\sqrt{y}}\,L\,(u,y)\,dy\right)du\end{aligned} \qquad (2.24)$$

Simplify integral equation (2.24) , we obtain [12,13]

$$\psi\,(x)-\frac{2}{\pi}\,x\int_0^1 (x/t)^{1/2}K\,(x,t)\,\psi(t)\,dt=F(x)\,, \qquad (2.25)$$

With kernel and free term respectively

$$K\,(x,t)\ =\int_0^\infty \frac{K_0\,(\alpha\,y)}{I_0\,(\alpha\,y)}\,I_{-1/2}\,(ty)\,I_{-1/2}\,(xy)\,y\,dy\,,$$

$$F(x)=\frac{2}{\sqrt{\pi}}\,x\frac{d}{dx}\int_0^x \frac{uf(u)}{\sqrt{x^2-u^2}}\,du\,.$$

Numerical techniques very effect tool to solve integral equation (2.25) for some specific known function $f(r)$ by using some software packages for example (mathematica or matlab) [2,9], moreover the free term and the kernel of (2.25) should be satisfied the inequalities

$$\int_0^1 |F(r)|\ dr<\infty\ ,\qquad \int_0^1\int_0^1 K^2(r,y)\,dr\,dy<\infty\ .$$

Notice that as $R\to\infty$, DSE (2.8),(2.9) were reduced to dual integral equations of the form

$$\int_0^\infty A(\lambda)\,J_0(\lambda r)\ d\lambda = f_1(r,s),\ \ 0<r<r_0\,,$$

$$\int_0^\infty \sqrt{\lambda^2+s/a}\ \ A(\lambda)\,J_0(\lambda r)\ d\lambda=0\ \ r_0<r<\infty\,.$$

The solution of the above dual integral equation discussed with details in[4,7]. The dual series method which is discussed above is an effect tool to investigate solutions of several problems with finite domain related to mixed boundary value problems involving heat and Helmholtz equations involving to different coordinate systems with many areas of physical and engineering applications.

3 Conclusion

The obtained results the conclusion can be drown that the paper aim an analytical method the mixed boundary value problem with boundary conditions of the first and of the second kind will lead to study dual series equations. The dual series equations always reduce to inhomogeneous Fredholm integral equation of the first kind. It can be revealed that iterations technique and numerical evaluation is an appropriate tools for solving

Fredholm integral equation of the first and of the second kinds. The present results can be served other investigations mixed problems, in particular for mixed boundary condition of the third kind with various applications and coordinate systems.

4 Open Problem

Several mixed initial boundary value problems deal with cylindrical and other coordinates still without solution, available known methods which is widely used to solve more simple mixed problems, sometimes difficult to use, for example we consider a mixed boundary conditions on the level surface of the cylinder $z=0,\ 0<r_0<r_1<R$ such that

$$\theta(r,0,\tau)=f_1(r,\tau), \quad 0\le r<r_0, \tag{2.26}$$

$$\theta_z(r,0,\tau)=f_2(\mathrm{r},\tau), \quad r_0<r<r_1 \tag{2.27}$$

$$\theta(r,0,\tau)=f_3(r,\tau), \quad r_0\le r<R \tag{2.28}$$

On the axis of the cylinder $r=R$, $r=0$, $z\to\infty$, we consider the same boundary conditions are (2.4)-(2.6) which is discussed in this paper with constant initial condition. Apply the mixed boundary conditions (2.26)-(2.28), in L-transform transform image, we obtain the following triple series equations

$$\sum_{n=1}^{\infty}\overline{C}_n(\lambda_n,s)J_0(\lambda_n\,\mathrm{r})=\overline{f_1}(\mathrm{r},s), \quad 0\le r<r_0 \ ,$$

$$\sum_{n=1}^{\infty}\sqrt{\lambda_n^2+s/a}\ \overline{C}_n(\lambda_n,s)\ J_0(\lambda_n\,\mathrm{r})=\overline{f_2}(\mathrm{r},s), \ r_0\le r<r_1 \ ,$$

$$\sum_{n=1}^{\infty}\overline{C}_n(\lambda_n,s)J_0(\lambda_n\,\mathrm{r})=\overline{f_3}(\mathrm{r},s), \quad r_1\le r<R.$$

The above triple equations may be solved by transformed these equations into a system of dual series equations.

References

[1] R. Coita, The integral transform method in thermal and fluids sciences and engineering . New York, Beqell House, Inc.1998.

[2] W. Hackbusch, *Integral* equation, Birkhäuser Verlag, Boston, 1995.

[3] N. Hoshan, The dual integral equations method to solve heat conduction equation for unbounded plate, Comp. Math and Modeling , 2010,v1,21, n2.

[4] N. Hoshan, Exact solution of certain dual integral equations involving heat equation F. East Journal of Applied Math, 2009, V 35, issue1, pp81-88. Pupsha.

[5] N. Hoshan, The dual integral equations method involving heat equation with mixed boundary conditions, Engineering Mathematics Letters, 2013, v2,137-142.

[6] N. Hoshan The dual integral equations method for solving Hlmholtz mixed boundary value problem, American Journal of computational and applied mathematics, 2013 , 3(2):138-142.

[7] N. Hoshan, Solution of Fredholm integral equations of the first kind, Applied mathematical sciences,2013, v.7, no 77,3847-3852.

[8] S. Indrano, A. Rmm, An iterative method for solving Fredolm integral equations of the first kind, Int. Computing sciences and math, 2009., v2, no.4.

[9] E. Kindal, , P. Atkinson ,Numerical solution of integral equations of the second kind, Cambridge University , 1997.

[10] B. Mandal , Advances in dual integral equations, London CRC. 1999 .

[11] I. Gradstien, I. Ryznik, Tables of integrals, series an product, Academic Press, New York 2000.

[12] I. Sneddon, Mixed boundary value problems, Amesterdam,1966.

[13] J. Uflyand , Dual equations in mathematical physics equations, L, Nauka, 1977.

[14] A. Verlan, V. Sizikov, Integral equations, Kiev ,Nauka Dumka,1986.