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Some growth properties of entire functions depending on their relative orders

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Abstract

In this paper we study some comparative growth properties of composite entire functions in terms of their maximum terms on the basis of their relative orders (relative lower orders) with respect to another entire function.

Keywords: *Entire function, maximum term, composition, relative order (relative lower order), growth. Entire function, maximum term, composition, relative order (relative lower order), growth.*

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1 Introduction, Definitions and Notations

Let $\mathbb{C}$ be the set of all finite complex numbers and f be an entire function defined on $\mathbb{C}$. The maximum modulus $M_f(r)$ of f on $|z| = r$ is defined as

$$M_f(r) = \max_{|z|=r} |f(z)| \ .$$

On the other hand, the maximum term $\mu_f(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ is defined by

$$\mu_f(r) = \max_{n \geq 0} (|a_n| r^n) \ .$$

We use the standard notations and definitions in the theory of entire functions which are available in [8]. In the sequel we use the following notation:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right), k = 1, 2, 3, ... and \ \log^{[0]} x = x.$$

If f is non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f^{-1}(r) : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s \to \infty} M_f^{-1}(s) = \infty$. Bernal [1] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf \{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0.\} \\ &= \limsup_{r \to \infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \end{aligned}$$

Similarly, one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r \to \infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

If we consider $g(z) = \exp z$, the above definition coincides with the classical definition { cf. [7] } of order (lower order) of an entire function f which is as follows:

Definition 1. *The order ρ_f and the lower order λ_f of an entire function f are defined as*

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r} \ and \ \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

Using the inequalities $\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f)$ $\{cf.\ [6]\}$, for $0 \leq r < R$ one may give an alternative definition of order(lower order) of entire function in the following manner:

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r}.$$

In the case of relative order, it therefore seems reasonable to state suitably an alternative definition of relative order of entire function in terms of its maximum terms. Datta and Maji [2] introduced such a definition in the following way:

Definition 2. *[2] The relative order $\rho_g(f)$ and the relative lower order $\lambda_g(f)$ of an entire function f with respect to another entire function g are defined as follows:*

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r} \text{ and } \lambda_g(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r}.$$

In this paper we wish to establish some results relating to the growth rates of composite entire functions in terms of their maximum terms on the basis of relative order (relative lower order).

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[5] Let f and g be any two entire functions Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f \circ g}(r) \leq \frac{\alpha}{\alpha - 1}\mu_f\left(\frac{\alpha R}{R-r}\mu_g(R)\right).$$

Lemma 2. *[5] If f and g are any two entire functions with $g(0) = 0$. Then for all sufficiently large values of r,*

$$\mu_{f \circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right)\right).$$

Lemma 3. *[2] If f be an entire function and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r,*

$$\mu_f(\alpha r) \geq \beta\mu_f(r).$$

3 Theorems

In this section we present the main results of the paper.

Theorem 4. *Let f, g and h be any three entire functions such that*

$$(i)\ \limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_g(r)\right)}{(\log r)^{\alpha}}=A,\ \text{a real number } >0,$$

$$(ii)\ \liminf_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_f(r)\right)}{\left(\log\mu_h^{-1}(r)\right)^{\beta+1}}=B,\ \text{a real number } >0$$

and $g(0)=0$ *for any* α,β *satisfying* $0<\alpha<1,\ \beta>0$ *and* $\alpha(\beta+1)>1$ *Then*

$$\rho_h(f\circ g)=\infty\ .$$

Proof. From (i) we have for a sequence of values of r tending to infinity,

$$\log\mu_h^{-1}\left(\mu_g(r)\right)\geq(A-\varepsilon)(\log r)^{\alpha} \tag{1}$$

and from (ii) we obtain for all sufficiently large values of r that

$$\log\mu_h^{-1}\left(\mu_f(r)\right)\geq(B-\varepsilon)\left(\log\mu_h^{-1}(r)\right)^{\beta+1}\ .$$

Since $\mu_g(r)$ is continuous, increasing and unbounded function of r, we get from above for all sufficiently large values of r that

$$\log\mu_h^{-1}\left(\mu_f(\mu_g(r))\right)\geq(B-\varepsilon)\left(\log\mu_h^{-1}\left(\mu_g(r)\right)\right)^{\beta+1}\ . \tag{2}$$

Also $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2, Lemma 3, (1) and (2) for a sequence of values of r tending to infinity that

$$\begin{aligned}
\log\mu_h^{-1}\mu_{f\circ g}(r) &\geq \log\mu_h^{-1}\left\{\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right\}\\
\text{i.e., }\log\mu_h^{-1}\mu_{f\circ g}(r) &\geq \log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}\\
\text{i.e., }\log\mu_h^{-1}\mu_{f\circ g}(r) &\geq (B-\varepsilon)\left(\log\mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)\right)^{\beta+1}\\
\text{i.e., }\log\mu_h^{-1}\mu_{f\circ g}(r) &\geq (B-\varepsilon)\left[(A-\varepsilon)\left(\log\left(\frac{r}{100}\right)\right)^{\alpha}\right]^{\beta+1}\\
\text{i.e., }\log\mu_h^{-1}\mu_{f\circ g}(r) &\geq (B-\varepsilon)(A-\varepsilon)^{\beta+1}(\log r+O(1))^{\alpha(\beta+1)}\\
\text{i.e., }\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r} &\geq \frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}(\log r+O(1))^{\alpha(\beta+1)}}{\log r}
\end{aligned}$$

$$\text{i.e., }\limsup_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}\geq\liminf_{r\to\infty}\frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}(\log r+O(1))^{\alpha(\beta+1)}}{\log r}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary and $\alpha\,(\beta+1)>1$, it follows from above that

$$\rho_h\,(f\circ g)=\infty,$$

which proves the theorem. $\square$

In the line of Theorem 4 one may state the following two theorems without their proofs :

Theorem 5. *Let f, g and h be any three entire functions such that*

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_g(r)\right)}{(\log r)^{\alpha}} &= A,\ \text{a real number } >0,\\ \limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_f(r)\right)}{\left(\log\mu_h^{-1}(r)\right)^{\beta+1}} &= B,\ \text{a real number } >0\end{aligned}$$

and $g\,(0)=0$ for any α,β satisfying $0<\alpha<1,\ \beta>0$ and $\alpha\,(\beta+1)>1$. Then

$$\rho_h\,(f\circ g)=\infty\ .$$

Theorem 6. *Let f, g and h be any three entire functions with*

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_g(r)\right)}{(\log r)^{\alpha}} &= A,\ \text{a real number } >0,\\ \liminf_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_f(r)\right)}{\left(\log\mu_h^{-1}(r)\right)^{\beta+1}} &= B,\ \text{a real number } >0\end{aligned}$$

and $g\,(0)=0$ for any α,β with $0<\alpha<1,\ \beta>0$ and $\alpha\,(\beta+1)>1$. Then

$$\lambda_h\,(f\circ g)=\infty\ .$$

Theorem 7. *Let f, g and h be any three entire functions satisfying*

$$\limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_g(r)\right)}{\left(\log^{[2]}r\right)^{\alpha}}=A,\ \text{a real number } >0,$$

$$\liminf_{r\to\infty}\frac{\log\left[\frac{\log\mu_h^{-1}\left(\mu_f(r)\right)}{\log\mu_h^{-1}(r)}\right]}{\left[\log\mu_h^{-1}(r)\right]^{\beta}}=B,\ \text{a real number } >0$$

and $g\,(0)=0$ for any α,β satisfying $\alpha>1,\ 0<\beta<1$ and $\alpha\beta>1$. Then

$$\rho_h\,(f\circ g)=\infty\ .$$

Proof. From (i) we have for a sequence of values of r tending to infinity we get that

$$\log \mu_h^{-1}\left(\mu_g(r)\right) \geq (A-\varepsilon)\left(\log^{[2]} r\right)^{\alpha} \tag{3}$$

and from (ii) we obtain for all sufficiently large values of r that

$$\begin{aligned} \log\left[\frac{\log \mu_h^{-1}\left(\mu_f(r)\right)}{\log \mu_h^{-1}(r)}\right] &\geq (B-\varepsilon)\left[\log \mu_h^{-1}(r)\right]^{\beta} \\ i.e., \ \frac{\log \mu_h^{-1}\left(\mu_f(r)\right)}{\log \mu_h^{-1}(r)} &\geq \exp\left[(B-\varepsilon)\left[\log \mu_h^{-1}(r)\right]^{\beta}\right] . \end{aligned}$$

Since $\mu_g(r)$ is continuous, increasing and unbounded function of r, we get from above for all sufficiently large values of r that

$$\frac{\log \mu_h^{-1}\left(\mu_f(\mu_g(r))\right)}{\log \mu_h^{-1}\left(\mu_g(r)\right)} \geq \exp\left[(B-\varepsilon)\left[\log \mu_h^{-1}\left(\mu_g(r)\right)\right]^{\beta}\right] . \tag{4}$$

Also $\mu_h^{-1}(r)$ is increasing function of r, it follows from (3), (4) , Lemma 2 and Lemma 3 for a sequence of values of r tending to infinity that

$$\begin{aligned} \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} &\geq \frac{\log \mu_h^{-1}\left\{\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right\}}{\log r} \\ i.e., \ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} &\geq \frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log r} \end{aligned}$$

$$\begin{aligned} & i.e., \ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} \\ \geq \ & \frac{\log \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)} \cdot \frac{\log \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}{\log r} \end{aligned}$$

$$\begin{aligned} & i.e., \ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} \\ \geq \ & \exp\left[(B-\varepsilon)\left[\log \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)\right]^{\beta}\right] \cdot \frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log r} \\ & i.e., \ \frac{\log \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} \\ \geq \ & \exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta}\right] \cdot \frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log r} \end{aligned}$$

$$
\begin{aligned}
& i.e., \ \frac{\log \mu_h^{-1} \mu_{f \circ g}(r)}{\log r} \\
\geq \ & \exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta-1} \log^{[2]}\left(\frac{r}{100}\right)\right] \cdot \frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log r} \\
& i.e., \ \frac{\log \mu_h^{-1} \mu_{f \circ g}(r)}{\log r} \\
\geq \ & \left(\log\left(\frac{r}{100}\right)\right)^{(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta-1}} \cdot \frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log r} \\
& i.e., \ \limsup_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{f \circ g}(r)}{\log r} \\
\geq \ & \liminf_{r \to \infty} \left(\log\left(\frac{r}{100}\right)\right)^{(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta-1}} \cdot \frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log r} .
\end{aligned}
$$

Since $\varepsilon\,(>0)$ is arbitrary and $\alpha > 1$, $\alpha\beta > 1$, the theorem follows from above. □

In the line of Theorem 7, one may also state the following two theorems without their proofs :

Theorem 8. *Let f, g and h be any three transcendental entire functions such that*

$$
\begin{aligned}
\liminf_{r \to \infty} \frac{\log \mu_h^{-1}\left(\mu_g(r)\right)}{\left(\log^{[2]} r\right)^{\alpha}} &= A, \ \textit{a real number} > 0, \\
\limsup_{r \to \infty} \frac{\log\left[\frac{\log \mu_h^{-1}\left(\mu_f(r)\right)}{\log \mu_h^{-1}(r)}\right]}{\left[\log \mu_h^{-1}(r)\right]^{\beta}} &= B, \ \textit{a real number} > 0
\end{aligned}
$$

and $g(0) = 0$ for any α, β with $\alpha > 1$, $0 < \beta < 1$ and $\alpha\beta > 1$. Then

$$\rho_h(f \circ g) = \infty \ .$$

Theorem 9. *Let f, g and h be any three transcendental entire functions such that*

$$
\begin{aligned}
\liminf_{r \to \infty} \frac{\log \mu_h^{-1}\left(\mu_g(r)\right)}{\left(\log^{[2]} r\right)^{\alpha}} &= A, \ \textit{a real number} > 0, \\
\liminf_{r \to \infty} \frac{\log\left[\frac{\log \mu_h^{-1}\left(\mu_f(r)\right)}{\log \mu_h^{-1}(r)}\right]}{\left[\log \mu_h^{-1}(r)\right]^{\beta}} &= B, \ \textit{a real number} > 0
\end{aligned}
$$

and $g(0)=0$ *for any* α,β *satisfying* $\alpha>1,\ 0<\beta<1$ *and* $\alpha\beta>1$. *Then*

$$\lambda_h(f\circ g)=\infty\ .$$

Theorem 10. *Let* f, g *and* h *be any three entire functions with* $0<\lambda_h(g)\le\rho_h(g)<\infty$, *and* $g(0)=0$ *and*

$$\limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_f(r)\right)}{\log\mu_h^{-1}(r)}=A,\ \ a\ real\ number\ <\infty.$$

Then

$$\lambda_h(f\circ g)\le A\lambda_h(g)\le\rho_h(f\circ g)\le A\rho_h(g)\ .$$

Proof. Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 1, Lemma 2 and Lemma 3 for all sufficiently large values of r that

$$\mu_h^{-1}\mu_{f\circ g}(r)\ge\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}\tag{5}$$

and

$$\mu_h^{-1}\mu_{f\circ g}(r)\le\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}\tag{6}$$

respectively.

Therefore from (5) we get for all sufficiently large values of r that

$$\log\mu_h^{-1}\mu_{f\circ g}(r)\ \ge\ \log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}$$

$$i.e.,\ \frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}\ \ge\ \frac{\log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log r}$$

$$i.e.,\ \frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$

$$\ge\ \frac{\log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log\mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}\cdot\frac{\log\mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}{\log r}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$

$$\ge\ \limsup_{r\to\infty}\left[\frac{\log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log\mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}\cdot\frac{\log\mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}{\log r}\right]$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$

$$\ge\ \limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log\mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}\liminf_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}{\log r}$$

$$i.e.,\ \ \rho_h\left(f\circ g\right)\geq A.\lambda_h\left(g\right)\ . \tag{7}$$

Similarly from (6), it follows for all sufficiently large values of r that

$$\log\mu_h^{-1}\mu_{f\circ g}(r) \leq \log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}$$
$$i.e.,\ \frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r} \leq \frac{\log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log r}$$

$$i.e.,\ \frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$
$$\leq \frac{\log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log\mu_h^{-1}\left(\mu_g\left(26r\right)\right)}\cdot\frac{\log\mu_h^{-1}\left(\mu_g\left(26r\right)\right)}{\log r} \tag{8}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$
$$\leq \liminf_{r\to\infty}\left[\frac{\log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log\mu_h^{-1}\left(\mu_g\left(26r\right)\right)}\cdot\frac{\log\mu_h^{-1}\left(\mu_g\left(26r\right)\right)}{\log r}\right]$$
$$i.e.,\ \liminf_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$
$$\leq \limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log\mu_h^{-1}\left(\mu_g\left(26r\right)\right)}\cdot\liminf_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_g\left(26r\right)\right)}{\log r}$$

$$i.e.,\ \ \lambda_h\left(f\circ g\right)\leq A.\lambda_h\left(g\right)\ . \tag{9}$$

Also from (8) we obtain for all sufficiently large values of r that

$$\limsup_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$
$$\leq \limsup_{r\to\infty}\left[\frac{\log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log\mu_h^{-1}\left(\mu_g\left(26r\right)\right)}\cdot\frac{\log\mu_h^{-1}\left(\mu_g\left(26r\right)\right)}{\log r}\right]$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$
$$\leq \limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log\mu_h^{-1}\left(\mu_g\left(26r\right)\right)}\cdot\limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_g\left(26r\right)\right)}{\log r}$$

$$i.e.,\ \ \rho_h\left(f\circ g\right)\leq A.\rho_h\left(g\right)\ . \tag{10}$$

Therefore the theorem follows from (7), (9) and (10) . □

Theorem 11. *Let f, g and h be any three entire functions satisfying $g(0)=0$ and $0<\lambda_h(g)\leq\rho_h(g)<\infty$ and*

$$\liminf_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_f(r)\right)}{\log\mu_h^{-1}(r)}=A,\ \text{a real number }<\infty.$$

Then

$$\lambda_h(f\circ g)\leq A\rho_h(g)\leq\rho_h(f\circ g)\ .$$

The proof of Theorem 11 is omitted because it can be carried out in the line of Theorem 10.

Theorem 12. *Let f, g and h be any three entire functions such that $g(0)=0$, $\rho_h(f)>0$ and $\lambda_g>0$. Then*

$$\limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log\mu_h^{-1}\left(\mu_f(r)\right)}=\infty\ .$$

Proof. Suppose $\rho_h(f)>0$ and $\lambda_g>0$.
As $\mu_h^{-1}(r)$ is an increasing function of r, we get from Lemma 2, for all sufficiently large values of r that

$$\begin{aligned}\log\mu_h^{-1}\mu_{f\circ g}(r) &\geq \log\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}\\ \text{i.e., }\log\mu_h^{-1}\mu_{f\circ g}(r) &\geq (\rho_h(f)-\varepsilon)\log M_g\left(\frac{r}{100}\right)\\ \text{i.e., }\log\mu_h^{-1}\mu_{f\circ g}(r) &\geq (\rho_h(f)-\varepsilon)\log M_g\left(\frac{r}{100}\right)\end{aligned}$$

$$\text{i.e., }\log\mu_h^{-1}\mu_{f\circ g}(r)\geq(\rho_h(f)-\varepsilon)\cdot\left(\frac{r}{100}\right)^{(\lambda_g-\varepsilon)}$$

$$\text{i.e., }\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}\geq\frac{(\rho_h(f)-\varepsilon)\cdot\left(\frac{r}{100}\right)^{(\lambda_g-\varepsilon)}}{\log r}$$

$$\text{i.e., }\limsup_{r\to\infty}\frac{\log\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}\geq\liminf_{r\to\infty}\frac{(\rho_h(f)-\varepsilon)\cdot\left(\frac{r}{100}\right)^{(\lambda_g-\varepsilon)}}{\log r}$$

$$\text{i.e., }\rho_h(f\circ g)=\infty\ . \tag{11}$$

Now in view of (11), we obtain that

$$\limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log\mu_h^{-1}\left(\mu_f(r)\right)}\ \geq\ \limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log r}\cdot\liminf_{r\to\infty}\frac{\log r}{\log\mu_h^{-1}\left(\mu_f(r)\right)}$$

$$\begin{aligned} i.e.,\ \limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log\mu_h^{-1}\left(\mu_f(r)\right)} &\geq \rho_h\left(f\circ g\right)\cdot\frac{1}{\rho_h\left(f\right)} \\ i.e.,\ \limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log\mu_h^{-1}\left(\mu_f(r)\right)} &= \infty\ . \end{aligned}$$

Thus the theorem follows. □

Theorem 13. *Let f, g and h be any three entire functions satisfying $g\left(0\right)=0$, $\lambda_h\left(f\right)>0$ and $\rho_g>0$. Then*

$$\limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log\mu_h^{-1}\left(\mu_f(r)\right)}=\infty\ .$$

Theorem 14. *Let f, g and h be any three entire functions such that $g\left(0\right)=0$, $\lambda_h\left(f\right)>0$ and $\lambda_g>0$. Then*

$$\limsup_{r\to\infty}\frac{\log\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log\mu_h^{-1}\left(\mu_f(r)\right)}=\infty\ .$$

The proofs of Theorem 13 and Theorem 14 are omitted as those can be carried out in the line of Theorem 12.

4 Open Problem

Actually this paper deals with the works on the growth properties of composite entire functions in terms of their maximum terms on the basis of their relative orders (relative lower orders) with respect to another entire function. Further, in order to determine the relative growth of two entire functions having same non zero finite relative order with respect to another entire function, Roy [4] introduced the definition of relative type of an entire function f with respect to another entire function g denoted as $\sigma_g\left(f\right)$ having non zero finite relative order $\rho_g\left(f\right)$ in the following way:

$$\sigma_g\left(f\right)=\limsup_{r\to\infty}\frac{M_g^{-1}M_f\left(r\right)}{r^{\rho_g(f)}}\ .$$

On the other hand, Datta and Biswas [3] introduced the definition of relative weak type of an entire function f with respect to another entire function g of finite positive relative lower order $\lambda_g\left(f\right)$ which is as follows:

$$\tau_g\left(f\right)=\liminf_{r\to\infty}\frac{M_g^{-1}M_f\left(r\right)}{r^{\lambda_g(f)}}\ .$$

Therefore using these two different relative growth indicators one may revisit the above growth estimations of composite entire functions under some different conditions. In this connection, the following natural questions may also be arisen :
1. Can these theories be modified by the treatment of the notions of relative order (respectively relative lower order), relative type and relative weak type of meromorphic functions?
2. Further can some extensions of the same be done for differential polynomials especially for wronskians and also for differential monomials?

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