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Certain Results on Starlike and Parabolic Starlike Functions

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Abstract

Using the technique of differential subordination, we obtain certain results for starlike and parabolic starlike functions involving the differential operator $\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)$. We study the above operator for multivalent functions and also conclude some results for univalent functions.

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1 Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N}; z \in \mathbb{E})$$

which are analytic and p-valent in the open unit disk $\mathbb{E}$. Obviously, $\mathcal{A}_1 = \mathcal{A}$, the class of all analytic functions f, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let the functions f and g be analytic in $\mathbb{E} = \{z : |z| < 1\}$. We say that f is subordinate to g in $\mathbb{E}$ (written as $f \prec g$,) if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1, \phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)), \ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1). The best dominant is unique up to a rotation of $\mathbb{E}$.
A function $f \in \mathcal{A}_p$ is said to be p-valent starlike of order $\alpha(0 \leq \alpha < p)$ in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \mathbb{E}. \tag{2}$$

We denote by $\mathcal{S}_p^*(\alpha)$, the class of p-valent starlike functions of order α. Note that $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$, which is the class of p-valent starlike functions.
A function $f \in \mathcal{A}_p$ is said to be p-valent parabolic starlike in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - p\right|, z \in \mathbb{E}. \tag{3}$$

We denote by $\mathcal{S}_{\mathcal{P}}^p$, the class of p-valent parabolic starlike functions. Note that $\mathcal{S}_{\mathcal{P}}^1 = \mathcal{S}_{\mathcal{P}}$, the class of parabolic starlike functions. Define the parabolic domain Ω as under

$$\Omega = \{u + iv : u > \sqrt{(u-p)^2 + v^2}\}.$$

Clearly the function

$$q(z) = p + \frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

maps the unit disk $\mathbb{E}$ onto the domain Ω. Hence the condition (3) is equivalent to

$$\frac{zf'(z)}{f(z)} \prec q(z), z \in \mathbb{E},$$

where q(z) is given above. In the literature of univalent function theory, the differential operator
$\frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right)$ is investigated by many authors for obtaining starlikenes of analytic functions. We refer to Lewandowski et al. [8], Ramesha et al. [1], Obradovič et al. [4], Padmanabhan [3], Li and Owa [2], Ravichandran et al. [6],[7]. In 1976, Lewandowski et al. [8] proved the following result:

Theorem 1.1 *If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > 0, z \in \mathbb{E}, \tag{4}$$

then $f \in \mathcal{S}^*$.

In 1995, Ramesha et al. [1] gave the following sufficient condition for starlikeness:

Theorem 1.2 *If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\} > 0, z \in \mathbb{E}, for\ some\ \alpha \geq 0, \tag{5}$$

then $f \in \mathcal{S}^*$.

Later on, Li and Owa [2] improved the above results and gave the following sufficient conditions for starlikeness:

Theorem 1.3 *If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\} > \frac{-\alpha}{2}, z \in \mathbb{E}, for\ some\ \alpha \geq 0, \tag{6}$$

then $f \in \mathcal{S}^*$.

Theorem 1.4 *If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\} > \frac{-\alpha^2(1-\alpha)}{4}, z \in \mathbb{E}, for\ some\ (0 \leq \alpha < 2), \tag{7}$$

then $f \in \mathcal{S}^*(\alpha/2)$.

Later on, Ravichandran et al. [6] proved the following result:

Theorem 1.5 *If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha\beta\left[\beta-\frac{1}{2}\right]+\left[\beta-\frac{\alpha}{2}\right], z \in \mathbb{E}, \alpha \geq 0, 0 \leq \beta \leq 1, \tag{8}$$

then $f \in \mathcal{S}^*(\beta)$.

In the present paper, we obtain certain results pertaining parabolic starlikeness in terms of the above differential operator as well as certain results for starlikeness.

2 Preliminaries

To prove our main results, we shall use the following lemma of Miller and Mocanu [5].

Lemma 2.1 *Let q be a univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q(z)$ and suppose that either*
(i) h is convex, or
(ii) Q is starlike.
In addition, assume that
(iii) $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ for all z in $\mathbb{E}$.
If p is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)], z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and q is the best dominant.

3 Main results

Theorem 3.1 *Let α be a non zero complex number and let $q(z)$ be a univalent convex function such that*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} + 2pq(z) + \frac{1-\alpha}{\alpha}\right) > 0. \tag{9}$$

If $f \in \mathcal{A}_p$ satisfies

$$\frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)pq(z) + \alpha p^2(q(z))^2 + \alpha pzq'(z), z \in \mathbb{E}, \tag{10}$$

then

$$\frac{zf'(z)}{pf(z)} \prec q(z), z \in \mathbb{E}.$$

Proof. Write $\dfrac{zf'(z)}{pf(z)} = u(z)$, in (10), we obtain:

$$(1-\alpha)pu(z) + \alpha p^2(u(z))^2 + \alpha pzu'(z) \prec (1-\alpha)pq(z) + \alpha p^2(q(z))^2 + \alpha pzq'(z)$$

Define $\theta(w) = (1-\alpha)pw + \alpha p^2w^2$ and $\phi(w) = \alpha p$. Therefore,

$$Q(z) = \phi(q(z))zq'(z) = \alpha pzq'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = (1-\alpha)pq(z) + \alpha p^2(q(z))^2 + \alpha pzq'(z)$$

On differentiating, we obtain $\dfrac{zQ'(z)}{Q(z)} = 1 + \dfrac{zq''(z)}{q'(z)}$ and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + 2pq(z) + \frac{1-\alpha}{\alpha}$$

In view of the given conditions, we see that Q is starlike and $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$. Therefore, the proof, now, follows from Lemma 2.1.

4 Applications with dominant $q(z) = \frac{1+(1-2\beta)z}{1-z}, \quad 0 \le \beta < 1, z \in \mathbb{E}.$

Remark 4.1 *Selecting* $q(z) = \dfrac{1+(1-2\beta)z}{1-z}$, *for* $0 \le \beta < 1$ *in Theorem 3.1. A little calculation yields:*

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{1-z}$$

and

$$1 + \frac{zq''(z)}{q'(z)} + 2pq(z) + \frac{1-\alpha}{\alpha} = \frac{2z}{1-z} + 2p\left(\frac{1+(1-2\beta)z}{1-z}\right) + \frac{1}{\alpha}$$

Obviously, for $\alpha > 0$, *q(z) is convex and satisfies the condition (9). Consequently, we get the following result.*

Theorem 4.1 *For a positive real number* α, *if* $f \in \mathcal{A}_p$ *satisfies*

$$\frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)p\left\{\frac{1+(1-2\beta)z}{1-z}\right\} + \alpha p^2\left\{\frac{1+(1-2\beta)z}{1-z}\right\}^2 + \frac{2\alpha pz(1-\beta)}{(1-z)^2}, 0 \le \beta < 1, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec p\left[\frac{1+(1-2\beta)z}{1-z}\right] \quad i.e.\ f \in \mathcal{S}_p^*(\gamma),\ where\ \gamma = \beta p\ and\ 0 \le \gamma < p.$$

Theorem 4.2 *If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)\right\} > (1-\alpha)p\beta+\alpha p^2\beta^2-\frac{1}{2}\alpha p(1-\beta), z \in \mathbb{E}, \alpha > 0, 0 \leq \beta < 1 \tag{11}$$

then $f \in S_p^*(\gamma)$, where $\gamma = \beta p$ and $0 \leq \gamma < p$.

By taking $\alpha = 1$ in Theorem 4.1, we conclude the following result.

Corollary 4.3 *If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec\ p^2\left\{\frac{1+(1-2\beta)z}{1-z}\right\}^2 + \frac{2pz(1-\beta)}{(1-z)^2}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec p\left[\frac{1+(1-2\beta)z}{1-z}\right]\ \textit{i.e. } f \in \mathcal{S}_p^*(\gamma),\ \textit{where } \gamma = \beta p \textit{ and } 0 \leq \gamma < p.$$

By taking $p = 1$ in Theorem 4.1, we get the following result for univalent starlike functions.

Corollary 4.4 *Suppose α is a positive real number and if $f \in \mathcal{A}$ satisfies the condition*

$$\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec\ (1-\alpha)\left\{\frac{1+(1-2\beta)z}{1-z}\right\} + \alpha\left\{\frac{1+(1-2\beta)z}{1-z}\right\}^2 + \frac{2\alpha z(1-\beta)}{(1-z)^2}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}\ \textit{i.e. } f \in \mathcal{S}^*(\beta), 0 \leq \beta < 1.$$

The Corollary 4.4 presents the following result for $\beta = 0$ and $\alpha = 1$.

Corollary 4.5 *If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec\ \left\{\frac{1+z}{1-z}\right\}^2 + \frac{2z}{(1-z)^2} = F(z), z \in \mathbb{E},$$

where $F(z)$ is a conformal mapping of the unit disk $\mathbb{E}$ and

$$F(\mathbb{E}) = \mathbb{C} \setminus \left\{w \in \mathbb{C} : \Re(w) \leq \frac{-1}{2}\right\}$$

then $f \in \mathcal{S}^*$.

Remark 4.2 *If we take p=1 and $\beta = 0$ in Theorem 4.2, then we obtain the result of Li and Owa [2] given in Theorem 1.3. For p=1, Theorem 4.2 reduces to result of Ravichandran [7] given in Theorem 1.5.*

5 Applications with dominant $q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, z \in \mathbb{E}$.

Remark 5.1 *When we select the dominant* $q(z) = 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ *in Theorem 3.1. A little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}$$

$$1 + \frac{zq''(z)}{q'(z)} + 2pq(z) + \frac{1-\alpha}{\alpha} = \frac{\sqrt{z}}{1-z}\left[\frac{3z-1}{2\sqrt{z}} + \frac{1}{\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}\right] + 2p\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right] + \frac{1}{\alpha}$$

Thus for positive real number α *we notice that q(z) is convex and condition (9) is satisfied by this dominant. Therefore, we, immediately arrive at the following result.*

Theorem 5.1 *Suppose* α *be a positive real number and if* $f \in \mathcal{A}_p$ *satisfies*

$$\frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)p\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)\right\} + \alpha p^2\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2 + \alpha pz\frac{4}{\pi^2\sqrt{z}(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right), z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec p + \frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad i.e.\ f \in \mathcal{S}^p_{\mathcal{P}}.$$

Keeping $\alpha = 1$ in Theorem 5.1, we have the following result.

Corollary 5.2 *If* $f \in \mathcal{A}_p$ *satisfies*

$$\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec p^2\left\{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2$$

$$+\frac{4pz}{\pi^2\sqrt{z}(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right), z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec p + \frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad i.e.\ f \in \mathcal{S}^p_{\mathcal{P}}.$$

Letting $p = 1$ in Theorem 5.1, we get the following result for parabolic starlikeness of univalent functions.

Corollary 5.3 *For a positive real number α if $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)\right\}$$

$$+\alpha\left\{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}^2$$

$$+\frac{4\alpha z}{\pi^2\sqrt{z}(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right), z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad i.e.\ f \in \mathcal{S}_{\mathcal{P}}.$$

6 Open Problem

The sufficient conditions for starlikeness and parabolic starlikeness of normalized analytic functions have been obtained in terms of the differential operator $\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right)$ only in case where $\alpha > 0$. The above conditions are still open for $\alpha < 0$. One may also investigate the above said operator for uniformly starlikeness and convexity of normalized analytic functions.

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