Int. J. Open Problems Complex Analysis, Vol. 15, No. 2, November 2023
ISSN 2074-2827;
www.i-csrs.org

The Characteristics of Inclusion Pertaining to Univalent Functions Associated with Bell Distribution Functions

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Received 2 June 2023; Accepted 5 October 2023

Abstract

In this study, we employ the new Bell Distribution (BD) operator to define a differential operator on an open unit disk $\mathbb{D}$. Additionally, this operator is introduced, having been chosen to be used in the process of applying the notion of differential subordination to the introduction of new subclasses of univalent functions. Coefficient estimates, growth and distortion theorems, closure theorems, and class-preserving integral operations for functions in the class.Furthermore, sufficient standards for starlike ness, convexity, and near-convexity for functions in the class The results of this study provide a novel technique for generating univalent functions using (BD) and provide an insightful understanding that will be helpful for future complicated analysis studies.

Keywords: univalent function, Bell distribution,inclusion relation, integral operator.

2010 Mathematical Subject Classification: 30C45.

1 preliminaries

Complex analysis includes a number of subfields, one of which is known as geometric function theory (GFT). This subfield attempts to explain the characteristics of the geometric qualities of the image domain of analytic functions. Because of this, we can define a geometric function as an analytic function that possesses particular geometric features.

Over the course of the years, various subclasses of the normalised analytic functions have been created and categorised.

Their features are being investigated from a wide variety of academic perspectives. There are a number of geometric features possessed by.

The use of analytical functions is discussed in a wide variety of authoritative sources, including [1, 2, 3, 4, 5, 6, 7].

Applications of probability distributions can be found in a wide variety of scientific domains, such as neural networks, economic forecasting, radiation-free sources, and meteorology, to name a few. In addition to this, a wide variety of real-world occurrences are modelled after them so that we can better understand them. This concept is put to good use in the study of many different aspects of mathematics, including orthogonal polynomials, derivatives of distributions, and singular structures of Laplacian eigenfunctions, to name just a few.

Aldawish et al. [8], and Amourah et al. [9] were the ones who initially published the Bell distribution, also abbreviated as the BD at times.

$$P\left(X=k\right)=\frac{q^{k}e^{e^{\left(-q^{2}\right)+1}}L_{k}}{k!};\quad k=1,2,3,...$$

where $L_k = \frac{1}{e}\sum\limits_{x=0}^{\infty}\frac{x^k}{k!}$ is the Bell numbers, $k \geq 2$, and $q > 0$.

The initial few digits on the Bell are $L_2 = 2$, $L_3 = 5$, $L_4 = 15$ and $L_5 = 52$.

Next, we'll supply a new power series, with Bell distribution probabilities serving as coefficients.

$$\mathbb{T}(q,z)=z+\sum_{k=2}^{\infty}\frac{q^{k-1}e^{e^{\left(-q^{2}\right)+1}}L_{k}}{(k-1)!}z^{k},\qquad z\in\mathbb{D},\ \text{where } q>0. \tag{1.1}$$

whose coefficients might be thought of as probabilities associated with the BD.

It is possible to demonstrate, through the use of the tried-and-true ratio test, that the aforementioned series is converging on unit disk $\mathbb{D}$. In later years, a number of academics made use of the BD of probability notion in order to achieve numerous aspects of the geometric functions theory in complex analysis. By utilizing the BD operator, Amourah et al. [10] were able to create a new subclass of bi-univalent functions.

Let's say that A stands for the classification of all analytical functions and that f is defined on the open unit disk. Conditions $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $f(0) = 0$ and $f\prime(0) - 1 = 0$, are required for this to be true. This results in an increase in each $f \in A$ form according to the Taylor series, which is as follows:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{D}). \tag{1.2}$$

If an analytical function $f(z) \in \mathcal{A}$ belongs to the class $\Re(t)$, which is composed of prestarlike functions of rank t in $\mathbb{D}$, then the class $\Re(t)$ contains it.

$$\frac{z}{(1-z)^{2(1-t)}} * f(z) \in \mathrm{S}_1^2, \quad t < 1.$$

If and only if a function $f(z) \in \mathcal{A}$ belongs to the class $\Re(0)$, then it can be said to be a member of the class C_1^2, which is composed of univalent convex functions. In addition to this, $\Re\,(0.5) = \mathrm{S}_1^2$ (see [11]). The investigation of inclusion relations of analytic functions in certain special sets is a topic that was once a part of the study of geometric function theory. Its roots can be traced back to the commencement of this field of study.

The neighbourhood and inclusion relations of univalent functions were the subject of research conducted by Ruscheweyh in [12]. The inclusion features of multivalent functions were looked into by Srivastava et al. [13]. In recent years, researchers have been looking into a number of different subclasses of univalent functions in the field of geometric function theory (for more information, see [14, 15, 16, 17, 18, 19]. Fuathermor, the emphasis of the attention of a large number of academics has shifted to the investigation of the links that exist between orthogonal polynomials and bi-univalent functions. In the event that you require any additional information, kindly refer to the sources listed in [20, 21, 22, 23, 24, 25, 26].

Let us consider the linear operator $\vartheta_q f(z) : \mathcal{A} \to \mathcal{A}$ as

$$\vartheta_q f(z) = \mathbb{T}(q,z) * f(z) = z + \sum_{k=2}^{\infty} \frac{q^{k-1} e^{e^{\left(-q^2\right)+1}} L_k}{(k-1)!} a_k z^k, \quad z \in \mathbb{D}.$$

For $Q \geq 0$, we define the operator $\vartheta_{q,Q}^m f(z) : \mathcal{A} \to \mathcal{A}$ as

$$\begin{aligned}
\vartheta_{q,Q}^0 f(z) =& z + \sum_{k=2}^{\infty} \frac{q^{k-1} e^{e^{\left(-q^2\right)+1}} L_k}{(k-1)!} a_k z^k, \\
\vartheta_{q,Q}^1 f(z) =& (1-Q)\vartheta_{q,Q}^0 f(z) + Qz \left(\vartheta_{q,Q}^0 f(z)\right)' \\
=& z + \sum_{k=2}^{\infty} \frac{q^{k-1} e^{e^{\left(-q^2\right)+1}} L_k}{(k-1)!} \left[1 + Q\left(k-1\right)\right] a_k z^k, \\
\vartheta_{q,Q}^2 f(z) =& (1-Q)\vartheta_{q,Q}^1 f(z) + Qz \left(\vartheta_{q,Q}^1 f(z)\right)' \\
=& z + \sum_{k=2}^{\infty} \frac{q^{k-1} e^{e^{\left(-q^2\right)+1}} L_k}{(k-1)!} \left[1 + Q\left(k-1\right)\right]^2 a_k z^k, \\
&\vdots \\
\vartheta_{q,Q}^m f(z) =& \quad (1-Q)\vartheta_{q,Q}^{m-1} f(z) + Qz \left(\vartheta_{q,Q}^{m-1} f(z)\right)' \\
\vartheta_{q,Q}^m f(z) =& z + \sum_{k=2}^{\infty} \frac{q^{k-1} e^{e^{\left(-q^2\right)+1}} L_k}{(k-1)!} \left[1 + Q\left(k-1\right)\right]^m a_k z^k, \\
& m \in \mathbb{N} \bigcup \{0\}, Q \geq 0, 0 < q \leq 1.
\end{aligned}$$

The class is specified by utilising the differential operator $\vartheta_{q,Q}^m$, which was provided for our assistance

$$\Omega_{\mathrm{m}}(l, Q, R)$$

As shown below: It is claimed that a function f in $\mathcal{A}$ belongs to the class $\Omega_{\mathrm{m}}(l, Q, R)$ if it meets the requirement that reads as follows:

$$\operatorname{Re}\left\{1 + \frac{1}{R}\left[(1-l)\frac{\vartheta_{q,Q}^m f(z)}{z} + l\left(\vartheta_{q,Q}^m f(z)\right)' - 1\right]\right\} > 0. \tag{1.3}$$

Or, equivalently:

$$\left|\frac{(1-l)\frac{\vartheta_{q,Q}^m f(z)}{z} + l\left(\vartheta_{q,Q}^m f(z)\right)' - 1}{(1-l)\frac{\vartheta_{q,Q}^m f(z)}{z} + l\left(\vartheta_{q,Q}^m f(z)\right)' - 1 + 2R}\right| < 1, \tag{1.4}$$

where $z \in D, l \geq 0, m \in \mathbb{N}_0$ and $R \in \mathbb{C} - \{0\}$.

Let us use the notation $\mathcal{A}^*$ to represent the subclass of $\mathcal{A}$ that consists of functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{1.5}$$

In addition, we will define the class $\Omega^*_{\mathrm{m}}(l, Q, R)$ using the following criteria:

$$\Omega^*_{\mathrm{m}}(l, Q, R) = \Omega_{\mathrm{m}}(l, Q, R) \cap \mathcal{A}^*. \tag{1.6}$$

In the paper we are about to present, we develop a number of interesting properties that can be applied to functions belonging to the class $\Omega^*_{\mathrm{m}}(l, Q, R)$. To our knowledge, no researcher has done this yet using any univalent functions related to Bell distribution functions.

2 Coefficient Inequalities

In this section, we determine the estimated values of the coefficients for the functions that belong to the class $\Omega^*_{\mathrm{m}}(l, Q, R)$. The most important theorem that we have developed for characterising this function class is referred to in the following reference as Theorem 2.1.

Theorem 2.1 *If and only if the conditions below are met, the function $f \in \mathcal{A}$ that is supplied by (1.2) is considered to be a member of the class $\Omega^*_{\mathrm{m}}(l, Q, R)$*

$$\sum_{k=2}^{\infty} [1 + l(k-1)] \frac{q^{k-1} e^{e^{\left(-q^2\right)+1}} L_k}{(k-1)!} \left[1 + Q\,(k-1)\right]^m a_k \leq |R|\,, \tag{2.1}$$

where $l \in \mathbb{C}, Q \geq 0, 0 < q \leq 1, k > 0$ and $m \in \mathbb{N} \cup \{0\}$

Proof. $f \in \Omega^*_{\mathrm{m}}(l, Q, R)$ is considered to be true if and only if the condition specified by reference number 1.4 is met. Assuming that $f \in \Omega^*_{\mathrm{m}}(l, Q, R)$, we have the following for $z \in D$:

$$\begin{aligned}
&\left|(1-l)\frac{\vartheta^m_{q,Q} f(z)}{z} + l\left(\vartheta^m_{q,Q} f(z)\right)' - 1\right| \\
&\quad - \left|(1-l)\frac{\vartheta^m_{q,Q} f(z)}{z} + l\left(\vartheta^m_{q,Q} f(z)\right)' - 1 + 2R\right| =
\end{aligned}$$

$$\left|\sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m a_k z^{k-1}\right|$$
$$-\left|2R+\sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m a_k z^{k-1}\right|$$
$$\leq\sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m a_k\left|z^{k-1}\right|-2|R|$$
$$-\sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m a_k\left|z^{k-1}\right|$$
$$\leq\sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m a_k-|R|\leq 0.$$

This implies

$$\sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m a_k\leq|R|.$$

On the other hand, if the condition outlined in the inequality (2.1),

$$\left|\frac{(1-l)\frac{\vartheta^m_{q,Q}f(z)}{z}+l\left(\vartheta^m_{q,Q}f(z)\right)'-1}{(1-l)\frac{\vartheta^m_{q,Q}f(z)}{z}+l\left(\vartheta^m_{q,Q}f(z)\right)'-1+2R}\right|<1.$$

The demonstration of Theorem 2.1 is now finished as a result of this.

Corollary 2.1 *If (1.2) is the source of $f\in\Omega^*_{\mathrm{m}}(l,Q,R)$, then the following holds true:*

$$a_k\leq\frac{|R|}{[1+l(k-1)]\frac{q^{k-1}e^{e^{(-q^2)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m}, k\geq 2.$$

3 Growth and Distortion Theorems

In order for function f to be included in the class, a growth and distortion attribute is required.

$$\Omega^*_{\mathrm{m}}(l, Q, R)$$

is contained in the following theorem.

Theorem 3.1 *If the function f that is defined by the reference number* 1.2 *belongs to the class $\Omega^*_{\mathrm{m}}(l, Q, R)$, then the following statement holds true: for the equation $|z| = r < 1$:*

$$|f(z)| \leq |r| + \frac{|R|}{[1+l(k-1)]^{\frac{q^{k-1}e^{e^{(-q^2)+1}}}{(k-1)!}L_k}\left[1+Q\left(k-1\right)\right]^m}\left|r\right|^k,$$

$$|f(z)| \geq r - \frac{|R|}{[1+l(k-1)]^{\frac{q^{k-1}e^{e^{(-q^2)+1}}}{(k-1)!}L_k}\left[1+Q\left(k-1\right)\right]^m}\left|r\right|^k.$$

Proof. The inequality may be easily derived from Theorem 2.1 thanks to the fact that $f \in \Omega^*_{\mathrm{m}}(l, Q, R)$.

$$\sum_{k=2}^{\infty} a_k \leq \frac{|b|}{[1+l(k-1)]^{\frac{q^{k-1}e^{e^{(-q^2)+1}}}{(k-1)!}L_k}\left[1+Q\left(k-1\right)\right]^m}. \tag{3.1}$$

We have Based on equation (1.2), we may express the following:

$$|f(z)| = \left|z + \sum_{k=2}^{\infty} a_k z^k\right| \leq |z| + \sum_{k=2}^{\infty} |a_k| \left|z^k\right|$$

That implies

$$|f(z)| \leq |z| + \sum_{k=2}^{\infty} |a_k| \left|z^k\right|$$

$$|f(z)| \leq |r| + \frac{|R|}{[1+l(k-1)]^{\frac{q^{k-1}e^{e^{(-q^2)+1}}}{(k-1)!}L_k}\left[1+Q\left(k-1\right)\right]^m}\left|r\right|^k.$$

Likewise, we can demonstrate that

$$|f(z)| \geq r - \frac{|R|}{[1+l(k-1)]^{\frac{q^{k-1}e^{e^{(-q^2)+1}}}{(k-1)!}L_k}\left[1+Q\left(k-1\right)\right]^m}\left|r\right|^k.$$

Theorem 3.2 *If the function f that is defined by the reference number* 1.2 *belongs to the class $\Omega^*_{\mathrm{m}}(l, Q, R)$, then the following statement holds true: for* *the equation* $|z| = r < 1$*:*

$$|f'(z)| \leq 1 + \frac{k|R|}{[1+l(k-1)]^{\frac{q^{k-1}e^{e^{(-q^2)+1}}L_k}{(k-1)!}}\left[1+Q\left(k-1\right)\right]^m}|r|^{k-1},$$

$$|f'(z)| \geq 1 - \frac{k|R|}{[1+l(k-1)]^{\frac{q^{k-1}e^{e^{(-q^2)+1}}L_k}{(k-1)!}}\left[1+Q\left(k-1\right)\right]^m}|r|^{k-1}.$$

Proof. The inequality may be easily derived from Theorem 2.1 thanks to the fact that $f \in \Omega^*_{\mathrm{m}}(l, Q, R)$.

$$\sum_{k=2}^{\infty} a_k \leq \frac{|b|}{[1+l(k-1)]^{\frac{q^{k-1}e^{e^{(-q^2)+1}}L_k}{(k-1)!}}\left[1+Q\left(k-1\right)\right]^m}.$$

Also

$$|f'(z)| = \left|1 + \sum_{k=2}^{\infty} k a_k z^{k-1}\right| \leq 1 + k + \sum_{k=2}^{\infty} |a_k| \left|z^{k-1}\right|$$

$$|f'(z)| =\leq 1 + k + \sum_{k=2}^{\infty} |a_k| \left|z^{k-1}\right|.$$

This is shows that

$$|f'(z)| \leq 1 + \frac{k|R|}{[1+l(k-1)]^{\frac{q^{k-1}e^{e^{(-q^2)+1}}L_k}{(k-1)!}}\left[1+Q\left(k-1\right)\right]^m}|r|^{k-1}.$$

Likewise, we can demonstrate that

$$|f(z)| \geq 1 - \frac{k|R|}{[1+l(k-1)]^{\frac{q^{k-1}e^{e^{(-q^2)+1}}L_k}{(k-1)!}}\left[1+Q\left(k-1\right)\right]^m}|r|^{k-1}.$$

4 Closure Theorems

Let the functions $f_b(z)$, $b = 1, 2, \cdots, I$, be defined by

$$f_b(z) = z - \sum_{k=2}^{\infty} a_{k,b} z^k, \quad a_{k,b} \geq 0 \ for \ z \in \mathbb{D}. \tag{4.1}$$

The theorems that are going to be presented give the closure theorems for the class $\Omega^*_{\mathrm{m}}(l, Q, R)$.

Theorem 4.1 *Allow the functions $f_b(z)$ that are defined by (4.1) to be members of the class.*

$$\Omega^*_{\mathrm{m}}(l,Q,R),$$

$l \in \mathbb{C}, Q \geq 0, 0 < q \leq 1, k > 0$ and $m \in \mathbb{N} \cup \{0\}$, for every $b = 1, 2, \cdots, I$. Then the function $G(z)$ defined by

$$G(z) = z - \sum_{\mathrm{k}=2}^{\infty} p_k z^{\mathrm{k}}, \quad p_k \geq 0 \tag{4.2}$$

possesses the property that it belongs to the class $\Omega^*_{\mathrm{m}}(l,Q,R)$, where

$$p_k = \frac{1}{I} \sum_{b=1}^{I} a_{k,b} (k \geq 2).$$

Proof. The conclusion that may be drawn from Theorem 2.1 is that, given that $f_b(z) \in \Omega_{\mathrm{m}}(l,Q,R)$, that

$$\sum_{k=2}^{\infty} [1 + l(k-1)] \frac{q^{k-1} e^{e^{\left(-q^2\right)+1}} L_k}{(k-1)!} \left[1 + Q\,(k-1)\right]^m a_{k,b} \leq |R|$$

for every $b = 1, 2, \cdots, I$. Henoe,

$$\begin{aligned}
&\sum_{k=2}^{\infty} [1 + l(k-1)] \frac{q^{k-1} e^{e^{\left(-q^2\right)+1}} L_k}{(k-1)!} \left[1 + Q\,(k-1)\right]^m p_k \\
&= \sum_{k=2}^{\infty} [1 + l(k-1)] \frac{q^{k-1} e^{e^{\left(-q^2\right)+1}} L_k}{(k-1)!} \left[1 + Q\,(k-1)\right]^m \left\{ \frac{1}{I} \sum_{j=1}^{I} a_{k,b} \right\} \\
&= \frac{1}{I} \sum_{b=1}^{I} \left(\sum_{k=2}^{\infty} [1 + l(k-1)] \frac{q^{k-1} e^{e^{\left(-q^2\right)+1}} L_k}{(k-1)!} \left[1 + Q\,(k-1)\right]^m a_{k,b} \right) \\
&\leq \frac{1}{I} \sum_{b=1}^{I} |R| = |R|
\end{aligned}$$

which implies that $G(z) \in \Omega^*_{\mathrm{m}}(l,Q,R)$.

Theorem 4.2 *Convex linear combination is used to classify the class denoted by the notation $\Omega^*_{\mathrm{m}}(l,Q,R)$,where $l \in \mathbb{C}, Q \geq 0, 0 < q \leq 1, k > 0$ and $m \in \mathbb{N} \cup \{0\}$*

Proof. Supposing that the functions $f_b(z)(b=1,2)$ defined by (4.1) belong to the class $\Omega_{\mathrm{m}}(l,Q,R)$, you can say that these functions belong to the class $\Omega_{\mathrm{m}}(l,Q,R)$. To demonstrate that the function exists, it is sufficient to provide evidence that

$$H(z) = \jmath f_1(z) + (1-\jmath)f_2(z) \quad (0 \leq \jmath \leq 1) \tag{4.3}$$

is also a member of the class $\Omega_{\mathrm{m}}(l,Q,R)$. Given that $0 \leq \jmath \leq 1$,

$$H(z) = z + \sum_{k=2}^{\infty} \left\{\jmath a_{k,1} + (1-\jmath)a_{k,2}\right\} z^k,$$

We have noticed that

$$\sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m \left\{\mathrm{J}a_{k,1}+(1-\mathrm{J})a_{k,2}\right\}$$
$$= \mathrm{J}\sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m a_{k,1}$$
$$+ (1-\mathrm{J})\sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m a_{k,2}$$
$$\leq \mathrm{J}|R| + (1-\mathrm{J})|R| = |R|.$$

Therefore, $H(z) \in \Omega_{\mathrm{m}}(l,Q,R)$. This concludes the demonstration of the validity of Theorem 4.2.

5 Integral Operators

Within the scope of this section, we will discuss integral transformations of functions within the class.

$$\Omega^*_{\mathrm{m}}(l,Q,R).$$

Theorem 5.1 *If the function f that is defined by the reference number* 1.5 *belongs to the class $\Omega^*_{\mathrm{m}}(l,Q,R)$, where $l \in \mathbb{C}, Q \geq 0, 0 < q \leq 1, k > 0$ and $m \in \mathbb{N} \cup \{0\}$. Consequently, the function $F(z)$, which is defined by*

$$F(z) = \frac{s+1}{z^e}\int_0^z T^{s-1}f(T)dT, \quad (s > -1) \tag{5.1}$$

likewise, it is a member of the class $\Omega^*_{\mathrm{m}}(l,Q,\alpha,\beta,R)$, too.

Proof. *From (5.1), it follows that* $F(z) = z - \sum_{k=2}^{\infty} h_k z^k$, *where* $h_k = \left(\frac{s+1}{s+k}\right) a_k$. *Therefore*

$$\begin{aligned}
&\sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m h_k \\
&= \sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m \left(\frac{s+1}{s+k}\right) a_k \\
&\leq \sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m a_k \leq |R|,
\end{aligned}$$

since $f(z) \in \Omega_{\mathrm{m}}^{*}(l, Q, R)$. *Hence by Theorem 2.1,* $F(z) \in \Omega_{\mathrm{m}}^{*}(l, Q, R)$.

6 Close-to-Convexity, Starlikeness and Convexity

If a function $f \in \mathcal{A}$ satisfies the conditions below, we say that it is close to convex order.

$$\operatorname{Re}\{f'(z)\} > \eta, \tag{6.1}$$

for some $\eta(0 \leq \eta \leq 1)$ as well as for all $z \in U$. In addition, a function $f \in \mathcal{A}$ is said to be starlike of order η if and only if it satisfies the following condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \eta, \tag{6.2}$$

for some $\eta(0 \leq \eta \leq 1)$ as well as for all $z \in U$. In addition, a function $f \in \mathcal{A}$ is said to be convex of order η if and only if $zf'(z)$ is starlike of order η, it satisfies the following condition:

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \eta, \tag{6.3}$$

for some $\eta(0 \leq \eta \leq 1)$ and for all $z \in D$.

Theorem 6.1 $f(z)$ *is close-to-convex of order* η *in* $|z| < h_1(k, l, R, \eta)$ *if and only if the condition* $f \in \Omega_{\mathrm{m}}^{*}(l, Q, R)$ *holds, where*

$$h_1(k,l,R,\eta) = \inf_k \left\{\frac{(1-\eta)[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m}{k|R|}\right\}^{\frac{1}{k-1}}.$$

Proof. The evidence presented is adequate to demonstrate that

$$|f'(z)-1| < \sum_{k=2}^{\infty} k a_k |z|^{k-1} \leq 1-\eta \tag{6.4}$$

and

$$\sum_{k=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}}{(k-1)!}L_k \left[1+Q\left(k-1\right)\right]^m a_k \leq |R|.$$

Consider the possibility that the statement (6.4) is accurate if you do so.

$$\frac{k|z|^{\mathrm{k}-1}}{1-\eta} \leq \frac{[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}}{(k-1)!}L_k \left[1+Q\left(k-1\right)\right]^m}{|R|}. \tag{6.5}$$

Using (6.5) to solve for $|z|$, we get the following:

$$|z| \leq \left\{\frac{(1-\eta)[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}}{(k-1)!}L_k \left[1+Q\left(k-1\right)\right]^m}{k|R|}\right\}^{\frac{1}{k-1}} \quad (k \geq 2).$$

Theorem 6.2 *$f(z)$ is starlike of order η in $|z| < h_2(k,l,R,\eta)$ if and only if the condition $f \in \Omega^*_{\mathrm{m}}(l,Q,R)$ holds, where*

$$h_2(k,l,R,\eta) = \inf_k \left\{\frac{(1-\eta)[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}}{(k-1)!}L_k \left[1+Q\left(k-1\right)\right]^m}{(k-\eta)|R|}\right\}^{\frac{1}{k-1}}.$$

Proof. That is something that needs to be demonstrated. $\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\eta$ for $|z| < h_2(k,l,R,\eta)$. Since

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k|z|^{k-1}}$$

Corollary 6.3 *$f(z)$ is conver of order η in $|z| < h_3(k,l,R,\eta)$ if and only if the condition $f \in \Omega^*_{\mathrm{m}}(l,Q,R)$ holds, where*

$$h_3(k,l,R,\eta)=\inf_k\left\{\frac{(1-\eta)[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m}{k(k-\eta)|R|}\right\}^{\frac{1}{k-1}}.$$

7 Inclusion properties

According to this theorem, the class $\Omega_{\mathrm{m}}(l,Q,R)$ has the following inclusion quality.

Theorem 7.1 *For Theorem 2.1, it is sufficient to satisfy the hypothesis. After that*

$$\Omega_{\mathrm{m}}(l_2,Q,R)\subseteq\Omega_{\mathrm{m}}(l_1,Q,R),$$

where

$$l_2\geq l_1\ .$$

Proof. Let $f\in\Omega_{\mathrm{m}}(l_2,Q,R)$. Thus, by applying Theorem 2.1, we are able to

$$\sum_{k=2}^{\infty}\left[1+l_2(k-1)\right]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m|a_k|\leq|R|$$

if $l_2\geq l_1$, implying that $1+l_2(k-1)\geq 1+l_1(k-1)$, in such that

$$\sum_{k=2}^{\infty}\left(1+l_2(k-1)\right)\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m\geq$$

$$\sum_{k=2}^{\infty}\left(1+l_1(k-1)\right)\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m.$$

This demonstrates that

$$\sum_{k=2}^{\infty}\left[1+l_1(k-1)\right]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m|a_k|\leq$$

$$|R|\leq\sum_{k=2}^{\infty}\left[1+l_2(k-1)\right]\frac{q^{k-1}e^{e^{\left(-q^2\right)+1}}L_k}{(k-1)!}\left[1+Q\left(k-1\right)\right]^m|a_k|\leq|R|$$

or

$$\sum_{n=2}^{\infty}[1+l(k-1)]\frac{q^{k-1}e^{e^{\left(-q^{2}\right)+1}}L_{k}}{(k-1)!}\left[1+Q\left(k-1\right)\right]^{m}a_{k}\leq\left|R\right|.$$

Therefore $f\in\Omega_{\mathrm{m}}(l_1,Q,R)$,this demonstrates that $f\in\Omega_{\mathrm{m}}(l_2,Q,R)\subseteq f\in\Omega_{\mathrm{m}}(l_1,Q,R)$.

8 conclusions

After defining a particular subclass of analytic univalent functions with the help of this new operator, we investigated a number of the class's geometric features. More notably, this class contains some new classes as well as some well-known ones, such as the class of convex functions and the class of starlike functions. Coefficient inequality, growth, distortion, integral operators, close-to-convexity, starlikeness, convexity, and a few closure criteria were among the attributes that were examined. Additionally, this study adds to our understanding by introducing the topic, examining its applications in GFT, and offering insights into the Some new ideas that may lead to more research have been revealed by the findings in this research report. Additionally, we have given researchers the chance to expand the use of this operator and provide fresh outcomes in both univalent as well as multivalent function theory. It should nonetheless be noted that the analysis of the subclass of analytic and univalent functions is the only focus of the findings in this work.

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